Magnetic monopoles, duality and supersymmetry

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1 Introduction

Note: Most of my treatment of the monopoles follows [9] closely.

1.1 Duality

Duality is the presence of two different perspectives of a single problem or of a single physical system. The main examples of the presence of duality would include the wave particle duality. A physical system can be viewed in the position space wave function (i.e. $\langle x|\psi\rangle$) and it can also be seen in the momentum space wave function (i.e. $\langle p|\psi\rangle$).
Another example for the concept of duality would be seen in the harmonic oscillator with the lagrangian

$$L = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 \tag{1.1}$$

It is easy to see that in the replacement

$$x \to \frac{p}{m\omega}, p \to -m\omega x \tag{1.2}$$

the harmonic oscillator is self dual.

2 Electromagnetic duality

2.1 Sourceless Maxwell equations

The sourceless Maxwell equations are given as follows (take $c = 1$)

$$\nabla.E = 0 \tag{2.1}$$

$$\nabla.B = 0 \tag{2.2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.3}$$

$$\nabla \times B = \frac{\partial E}{\partial t} \tag{2.4}$$

Now, it is easy to see that in following transformation

$$D : E_i \rightarrow B_i, B_i \rightarrow -E_i \tag{2.5}$$

(2.1) – (2.4) remain invariant. In fact, we can make this into a continuous transformation like;

$$E_i \rightarrow cos\theta E_i + sin\theta B_i; B_i \rightarrow -sin\theta E_i + cos\theta B_i \tag{2.6}$$

and it is straightforward to see that (2.1) – (2.4) are still invariant.
We can now introduce the electromagnetic field tensor $F_{\mu\nu}$ with

$$E_i = F_{0i}; B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{2.7}$$

Now, we can translate the D transformation into the terms of $F_{\mu\nu}$ tensor as follows;

$$E_i \rightarrow B_i \Rightarrow F_{0i} \rightarrow \frac{1}{2}\epsilon_{ijk}F_{jk} = *F_{0i}$$

$$B_i \rightarrow -E_i \Rightarrow \frac{1}{2}\epsilon_{ijk}F_{jk} \rightarrow -F_{0i} = \frac{1}{2}\epsilon_{ijk} * F_{jk}$$

where

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

Now, we can see that the D transformation can be seen as;

$$F_{\mu\nu} \rightarrow *F_{\mu\nu}$$

If there are no monopoles, we can take

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = (\nabla \times A)_i = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j)$$

and (as a reminder, I am using $(+ - --)$ signature)

$$E_i = F_{0i} = (-\nabla\phi - \frac{\partial A}{\partial t})_i = \partial_0 A_i - \partial_i A_0$$

So, we can conclude that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.8}$$

2.2 Electromagnetism in Quantum Mechanics

In Quantum Mechanics, the electromagnetic coupling can be realised by minimum coupling scheme

$$\hat{p}_j \rightarrow -i(\nabla - ieA)_j \tag{2.9}$$

and it is a well known fact that the Schrodinger equation is invariant in the gauge transformation,

$$\psi \rightarrow e^{-ie\chi}; A_i \rightarrow A_i - \frac{i}{e}e^{ie\chi}\nabla e^{-ie\chi} \tag{2.10}$$

where $e^{ie\chi}$ is an element of a $U(1)$ group.
In 1931, Dirac tried to add magnetic monopoles without disturbing the coupling [1]. His work gave a remarkable result which gave a reason for the quantization of electrical charge in the presence of magnetic monopoles. We can understand that work by considering a magnetic monopole at $r = 0$ and since we have a magnetic monopole now, it cant the case that $B_i = (\nabla \times A)_i$ everywhere. However, we can still define A_i in patches and in the region where the patches overlap, they should differ by a gauge transformation. So, far away from the monopole (i.e. say, for $r > r_0$) the magnetic field due to the monopole should be like

$$\mathbf{B} = \frac{g\hat{r}}{4\pi r^2} \tag{2.11}$$

Now,we want some $\mathbf{A}$ to give the above magnetic field by taking its curl. We use the polar coordinate system and in this system, the components of $\mathbf{B}$ for an arbitrary $\mathbf{A}$ are given as;

$$B_r = \frac{1}{rsin\theta}\left[\frac{\partial(sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$B_\theta = -\frac{1}{rsin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial(rsin\theta A_\phi)}{\partial r}\right]$$

$$B_\phi = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

Since we want the ϕ component of $\mathbf{B}$ to be zero, we set A_θ and A_r equal to zero. Moreover, for $\mathbf{B}$ to fall as $1/r^2$, the components should fall as $1/r$ and thus, rA_ϕ should be independent of r. This renders B_θ equal to zero and well. Then for B_r then, we have;

$$B_r = \frac{1}{rsin\theta}\frac{\partial(sin\theta A_\phi)}{\partial\theta} = \frac{g}{4\pi r^2} \Rightarrow A_\phi = \frac{g}{4\pi r}\frac{\xi - cos\theta}{sin\theta}$$

where ξ is arbitrary.
Now, we can choose the vector potential in two patches that cover the S^2 circle i.e. the Northern patch and the Southern patch and call the respective $\mathbf{A}$ as $\mathbf{A}_N$ and $\mathbf{A}_S$. For the northern patch, we choose $\xi = 1$ and for the southern patch, we choose $\xi = -1$. Then, we will have the following vector potentials;

$$\mathbf{A}_N = \frac{g}{4\pi r}\frac{1 - cos\theta}{sin\theta}\hat{\phi} \tag{2.12}$$

$$\mathbf{A}_S = -\frac{g}{4\pi r}\frac{1 + cos\theta}{sin\theta}\hat{\phi} \tag{2.13}$$

The overlap region happens to be $\theta = 90^0$ and the difference of the two potentials in this region is as follows;

$$A_N(\theta = \frac{\pi}{2}) - A_S(\theta = \frac{\pi}{2}) = \frac{g}{2\pi r}\hat{\phi} = -\nabla(-\frac{g}{2\pi}\phi) = -\nabla\chi$$

So, we see that the difference in the overlap zone is indeed a gauge transformation and thus, in the overlap zone, the two potentials are physically equivalent.
Now, the $U(1)$ element (i.e. $e^{ie\chi}$) should be continuous and thus, we have the following condition;

$$e^{ie[\chi(2\pi)-\chi(0)]} = 1 \Rightarrow e[\chi(2\pi) - \chi(0)] = 2\pi n \, (n \in \mathbb{Z}) \tag{2.14}$$

Using the defintion of $\chi(\phi)$, we have;

$$g = \chi(0) - \chi(2\pi) \tag{2.15}$$

Using (2.14) and (2.15), we get;

$$eg = 2\pi n \, (n \in \mathbb{Z}) \tag{2.16}$$

(2.16) is known as the Dirac quantization condition.
Some comments are in order. Firstly, this condition implies that the presence of a single monopole in the universe will imply the quantization of electrical charge.
Secondly, this quantization condition implies that $e\chi = 0$ and $e\chi = 2\pi$ actually represent the same $U(1)$ element and thus, the $U(1)$ group is compact. In other words, the presence of monopoles imply the presence of compact $U(1)$ group. Using the contrapositive of this statement we can say that the absence of compact $U(1)$ group will imply the absence of magnetic monopoles. The compact $U(1)$ group will be present in theories where a larger group symmetry has been spontaneously broken to a smaller group containing $U(1)$.
At small distances, the monopoles carry color magnetic charge and the combination of color magnetic charge and ordinary magnetic charge satisfy the Dirac quantization condition. More details in [2]

3 The t'Hooft, Polyakov monopole

In the Dirac's treatment of the monopole, the interior of the monopole is not studied. The long distance behaviour of the magnetic field is studied and the Dirac's quantization condition is derived. In order to study the interior of a monopole, we study a monopole configuration called t'Hooft, Polyakov monopole. It is nothing but a non abelian Yang Mills Higgs theory in $3+1$ dimensions. So, the lagrangian of the theory is as follows;

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) \, , \langle \phi \rangle = v \neq 0 \tag{3.1}$$

The equation of motion for $F^a_{\mu\nu}$ is (also called the Gauss constraint)

$$D_\mu F^{\mu\nu}_a = e \epsilon_{abc} \phi_b D^\nu \phi_c \tag{3.2}$$

Define $F_{\mu\nu}$ as;

$$F_{\mu\nu} = \hat{\phi}^a F^a_{\mu\nu} - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D^\mu \hat{\phi}^c$$

where $\hat{\phi}^a$ is a unit vector in the direction of ϕ^a and for this vector to exist, $\phi^a \neq 0$ anywhere.
Now, using the definition of the Yang Mills field $F^a_{\mu\nu}$, we can express $F_{\mu\nu}$ in terms of A^a_μ. The proof goes as follows.

$$F_{\mu\nu} = \hat{\phi}^a [\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e \epsilon^{abc} A^b_\mu A^c_\nu] - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a [\partial_\mu \hat{\phi}^b + e \epsilon^{bde} A^d_\mu \hat{\phi}^e][\partial_\nu \hat{\phi}^c + e \epsilon^{cfg} A^f_\nu \hat{\phi}^g]$$

$$= \partial_\mu (\hat{\phi}^a A^a_\nu) - \partial_\nu (\hat{\phi}^a A^a_\mu) - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a \partial_\mu \hat{\phi}^b \partial^\mu \hat{\phi}^c$$

$$+ A^d_\mu [\partial_\nu \hat{\phi}^a \delta^{ad} - \partial_\nu \hat{\phi}^c \epsilon^{abc} \epsilon^{bde} \hat{\phi}^a \hat{\phi}^e] - A^f_\nu [\partial_\mu \hat{\phi}^a \delta^{af} + \partial_\mu \hat{\phi}^b \epsilon^{abc} \epsilon^{cfg} \hat{\phi}^a \hat{\phi}^g] + e \epsilon^{abc} \hat{\phi}^a A^b_\mu A^c_\nu - e \epsilon^{abc} \hat{\phi}^a \epsilon^{bde} \epsilon c f g A^d_\mu A^f_\nu \hat{\phi}^e \hat{\phi}^g \tag{3.3}$$

In the equation above, the terms that we want are the first three terms. The terms in the second line vanish. It can be shown as follows.
Take the first bracketed term in (3.3). It is;

$$\partial_\nu \hat{\phi}^a \delta^{ad} - \partial_\nu \hat{\phi}^c \epsilon^{abc} \epsilon^{bde} \hat{\phi}^a \hat{\phi}^e = \partial_\nu \hat{\phi}^d + \partial_\nu \hat{\phi}^c (\delta^{ad}\delta^{ce} - \delta^{ae}\delta^{cd}) \hat{\phi}^a \hat{\phi}^e = \partial_\nu \hat{\phi}^d + \hat{\phi}^d \hat{\phi}^e \partial_\nu \hat{\phi}^e - \partial_\nu \hat{\phi}^d = 0$$

Where I used the fact that

$$\hat{\phi}^d \partial_\nu \hat{\phi}^d = \frac{1}{2}\partial_\nu (\hat{\phi}^d \hat{\phi}^d) = \frac{1}{2}\partial_\nu (1) = 0$$

The second bracketed term in the second line of (3.3) can be shown to vanish in a similar way. Now, we need to prove that the final two terms in (3.3) vanish. In other words, we need to show that;

$$\epsilon^{abc} \hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc} \hat{\phi}^a \epsilon^{bde} \epsilon^{cfg} A^d_\mu A^f_\nu \hat{\phi}^e \hat{\phi}^g = 0$$

We can manipulate the last term as

$$\epsilon^{abc} \hat{\phi}^a \epsilon^{bde} \epsilon^{cfg} A^d_\mu A^f_\nu \hat{\phi}^e \hat{\phi}^g = (\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf}) \hat{\phi}^a \epsilon^{bde} A^d_\mu A^f_\nu \hat{\phi}^e \hat{\phi}^g = \epsilon^{edb} \hat{\phi}^e A^d_\mu A^b_\nu$$

where I have dropped the vanishing term $\epsilon^{bde} \hat{\phi}^a \hat{\phi}^b \hat{\phi}^e A^d_\mu A^a_\nu$ and used the fact that $\hat{\phi}^e \hat{\phi}^e = 1$.
We can now see that;

$$\epsilon^{abc} \hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc} \hat{\phi}^a \epsilon^{bde} \epsilon^{cfg} A^d_\mu A^f_\nu \hat{\phi}^e \hat{\phi}^g = \epsilon^{abc} \hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc} \hat{\phi}^a A^b_\mu A^c_\nu = 0$$

So, we have shown that;

$$F_{\mu\nu} = \partial_\mu (\hat{\phi}^a A^a_\nu) - \partial_\nu (\hat{\phi}^a A^a_\mu) - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a \partial_\mu \hat{\phi}^b \partial^\mu \hat{\phi}^c \tag{3.4}$$

Now, a magnetic field can be defined as;

$$B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} = \epsilon_{ijk} \partial_j (\hat{\phi}^a A^a_k) - \frac{1}{2e} \epsilon_{ijk} \epsilon^{abc} \hat{\phi}^a \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c$$

Now, in order to calculate the magnetic flux over a surface, we need to integrate this magnetic field over a sphere. This will receive zero contribution from the first term as it is a curl and the sphere has no boundary. The second term gives the following contribution

$$g = \int_{S^2} dS_i B_i = \frac{1}{2ev^3} \int dS_i \epsilon_{ijk} \epsilon^{abc} \phi^a \partial_j \phi^b \partial_k \phi^c$$

This integral is evaluated as by choosing the form $\phi^a = v\hat{x}^a$ at infinity (where $x^a = r\hat{x}^a$). We simply get;

$$\begin{aligned} \frac{1}{v^3} \int_{S^2} dS_i \epsilon_{ijk} \epsilon_{abc} \phi^a \partial_j \phi^b \partial_k \phi^c &= \int_{S^2} r^2 d\Omega \hat{x}_i \epsilon_{ijk} \epsilon_{abc} \hat{x}_a \partial_j \hat{x}_b \partial_k \hat{x}_c \\ &= \int_{S^2} d\Omega \hat{x}_i \epsilon_{ijk} \epsilon_{abc} \hat{x}_a [\delta_{bj}\delta_{ck} - 2\delta_{bj\hat{x}_k\hat{x}_c}] \end{aligned} \tag{3.5}$$

Where I have dropped the vanishing term and used the identity;

$$\partial_i \hat{x}_j = \frac{1}{r}(\delta_{ij} - \hat{x}_i \hat{x}_j)$$

Now, using the identities;

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj},\ \epsilon_{ijk}\epsilon_{jkl} = 2\delta_{il}$$

the integral in (3.5) becomes;

$$2\int_{S^2} d\Omega \hat{x}_i \hat{x}_a [\delta_{ai} - \delta_{ai} + \hat{x}_a \hat{x}_i] = 8\pi \Rightarrow g = -\frac{4\pi}{e}$$

Now, I can choose another form of ϕ^a by multiplying the azimuthal angle φ by an integer n. In other words, I can choose ϕ^a to be

$$\phi^a(\theta, n\varphi) = v\hat{x}^a(\theta, \varphi)$$

This will simply multiply the above integral n times as we are going through S^2_ϕ n times. So, we get the following condition;

$$eg = 4\pi n,\ n \in \mathbb{Z} \tag{3.6}$$

This is the t'Hooft quantization condition.

4 Bogomol'nyi bound and BPS solution

4.1 Bogomol'nyi bound

We calculate the energy for the static solution of YMH theory in the gauge $A_0^a = 0$ and we get;

$$E = \int d^3x \left[\frac{1}{2}B_i^a B_i^a + \frac{1}{2}(D_i\phi^a)^2 + V(\phi)\right] = \int d^3x [\frac{1}{2}(B_i^a - D_i\phi^a)^2 + B_i^a D_i\phi^a + V(\phi)] \tag{4.1}$$

Now, using the zeroth component Bianchi identity $\epsilon^{\mu\nu\alpha\beta}D_\nu F^a_{\alpha\beta} = 0$ and using the definition of magnetic field in terms of the $F^a_{\mu\nu}$ tensor i.e. $2B_i^a = \epsilon_{ijk}F^a_{jk}$, we get;

$$\epsilon^{0ijk}D_i F^a_{jk} = D_i B_i^a = 0$$

Moreover, since $B_i^a\phi^a$ is gauge singlet, we have $D_i(B_i^a\phi^a) = \partial_i(B_i^a\phi^a)$ and thus, the second term in the integrand in (4.1) can be written as $\partial_i(B_i^a\phi^a)$ and thus, it is a total derivative which implies that the space integral can be done easily on it by converting it to a surface integral with the integrand $B_i^a\phi^a$. Using the result from the previous section, we can see that this integral is nothing but vg but in order to compensate negative magnetic charge, we write this as vg. Thus, (4.1) becomes;

$$E = vg + \int d^3x [\frac{1}{2}(B_i^a - D_i\phi^a)^2 + V(\phi)] \tag{4.2}$$

We can see that since the integral is positive definite, there will be a lower bound on the energy of the static solution and thus,

$$E \geq vg$$

This is also known as the Bogomol'nyi bound.
Note: I am doing all of this for positive magnetic charge. All of this can be done for negative magnetic charge as well.

4.2 BPS solution

If we impose the condition $B_i^a = D_i\phi^a$ (also known as the Bogomol'nyi equation) and set $V(\phi) = 0$ everywhere, then the energy of the field configuration is just vg. There is one thing that needs some mentioning here. Since we are setting $V(\phi) = 0$ everywhere, it seems as if we won't be able to get a nonzero $\langle\phi^a\phi^a\rangle$. However, since we need $\langle\phi^a\phi^a\rangle = v^2$ at spatial infinity, we can impose this as a boundary condition.
If we try to give A_i^a a non zero vev, it will break Lorentz invariance. Moreover, for the theory to have non zero topological charge, ϕ^a should vary at infinity and thus, solutions can't be rotationally invariant. However, we can make the solutions invariant under the diagonal subgroup $SO(3)_s \times SO(3)_g$ where $SO(3)_g$ is the global gauge group. We would not need the detailed form of the BPS solutions in this thesis though.

5 Moduli space of the BPS solution

Moduli space is the space of fixed energy and fixed topological charges solutions. The coordinates on the moduli space are called moduli or collective coordinates.
We can we easily see that if $\phi^a(x)$ is a solution, then $\phi^a(x+X)$ is also a solution due to Poincare invariance (where X is a fixed $3-$ vector). So, the moduli space of BPS solution contains $\mathbb{R}^3$ as a product.
We now consider the Gauss constraint (3.2) and expand it to get;

$$D_\mu[\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu] = e\epsilon_{abc}\phi_b[\partial^\nu\phi_c + e\epsilon_{cde}A_d^\nu\phi_e]$$

Setting $\nu = 0$ and working in the $A_a^0 = 0$ gauge, we get;

$$D_i\dot{A}_a^i + e[\phi, \dot{\phi}]_a = 0$$

The linearized form of the Gauss' law for deformations to the BPS solutions $(\delta\phi^a, \delta A_i^a)$ is obtained by letting $A_i^a \to A_i^a + \delta A_i^a, \phi^a \to \phi^a + \delta\phi^a$ and realising that only the deformations are allowed to the time dependant. We get;

$$D_i\delta\dot{A}_a^i + e[\phi, \delta\dot{\phi}]_a = 0$$
$$D_i\delta\dot{A}_a^i + e[\phi, \delta\dot{\phi}]_a = 0 \tag{5.1}$$

Moreover, we can also find the linearized form of the Bogomol'nyi equation by expanding it and then linearizing it as;

$$\epsilon_{ijk}[\partial^j\delta A_a^k + e\epsilon_{abc}A_b^j\delta A_c^k] = \partial_i\delta\phi^a + e\epsilon^{abc}A_i^b\delta\phi^c + e[\delta A_i, \phi]_a$$

This gives us;

$$\epsilon_{ijk}D^j\delta A_a^k = D_i\delta\phi_a + e[\delta A_i, \phi]_a \tag{5.2}$$

The solution to the equations $(5.1) - (5.2)$ are

$$\delta A_0^a = 0,\ \delta\phi^a = 0,\ \delta A_i^a = D_i(\xi(t)\phi^a)$$

where $\xi(t)$ is any arbitrary function of time.
We can see that if $\dot{\xi} = 0$, then the change in A_i^a is a large gauge transformation (i.e. it does not vanish at infinity) and the A_i^a fields which are connected by large gauge transformations are not physically equivalent (unlike small gauge transformations). So, the $U(1)$ element which corresponds to this gauge transformation is $e^{\xi\phi}$ and since the unbroken $U(1)$ group is compact, the coordinate ξ has to be compact and thus, the moduli space of BPS solutions is $\mathbb{R}^3 \times S^1$.
More details on the moduli space of BPS solution are given in [3].

6 Witten effect

We can add a theta term in the Yang Mills lagrangian without breaking the gauge invariance. This term is of the form;

$$\mathcal{L}_\theta = -\frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} * F^{\mu\nu}_a \tag{6.1}$$

The prefactor is chosen for the later convenience. We can study the effect of this term for monopoles in the simplest case first (i.e. the QED case with $U(1)$ gauge group).

6.1 QED case

For QED, we calculate $F^{\mu\nu} * F_{\mu\nu}$ first to get;

$$F^{\mu\nu} * F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \frac{1}{2}\left[\epsilon^{0ijk} F_{0i} F_{jk} + \epsilon^{i0jk} F_{i0} F_{jk} + \epsilon^{ij0k} F_{ij} F_{0k} + \epsilon^{ijk0} F_{ij} F_{k0}\right]$$

$$= 2\epsilon^{0ijk} F_{0i} F_{jk} = 4\mathbf{E}.\mathbf{B}$$

So, we get;

$$\mathcal{L}_\theta = -\frac{\theta e^2}{8\pi^2}\mathbf{E}.\mathbf{B}$$

For a static monopole, we have;

$$\mathbf{E} = \nabla A_0,\ \mathbf{B} = \nabla \times \mathbf{A} + \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r}$$

This leads to the following lagrangian;

$$L_\theta = \int \mathcal{L}_\theta = \frac{e^2 g\theta}{8\pi^2}\int d^3x A_0 \delta^{(3)}(\mathbf{r}) \tag{6.2}$$

where we used the result;

$$\mathbf{E}.\mathbf{B} = \nabla A_0.\left[\nabla \times A + \frac{g}{4\pi}\frac{\hat{r}}{r}\right] = -\frac{g}{4\pi} A_0 \nabla.\left(\frac{\hat{r}}{r}\right) + (\text{surface term}) = -gA_0\delta^{(3)}(\mathbf{r})$$

Now, using the quantization condition $eg = 4\pi$ for a single monopole, (6.2) becomes;

$$L_\theta = -\left[-\frac{e\theta}{2\pi}\int d^3x A_0 \delta^{(3)}(\mathbf{r})\right]$$

This is nothing but the interaction term for a charged particle located at the origin with a background field A_0 with charge;

$$-\frac{e\theta}{2\pi}$$

So, the theta term can give charge to the monopoles.

6.2 SU(2) case

Now, we can analyse a similar scenario in the $SU(2)$ theory. For this purpose, we consider the large gauge transformations that act on A_μ. The finite deformation (i.e. without the infinitesimal parameter) can be written as;

$$\delta A^a_\mu = \frac{1}{ev} D_\mu \phi^a$$

where v is written for the dimension to work out correctly (Remember that $[A_\mu] = 1$ and $[D_\mu] = 1$ where I am specifying the mass dimension). Let the generator of this gauge transformation be $\mathcal{N}$. Now, since we are not talking about the fermions right now and since the parameter of this large gauge transformation lives on S^1, we can see that $e^{2\pi i\mathcal{N}} = 1$. Since large gauge transformations act on the field, $\mathcal{N}$ is nothing but the conserved Noetherian current corresponding to the deformations of A^a_μ. It can be easily worked out as follows.

$$\mathcal{N} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_\mu)} \delta A^a_\mu$$

Now, we write the lagrangian as;

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} - \frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} * F^{\mu\nu}_a$$

So, we can calculate the required derivative as;

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_\mu)} = -\frac{1}{2} F^{\alpha\beta}_b \frac{\partial F^b_{\alpha\beta}}{\partial(\partial_0 A^a_\mu)} - \frac{\theta e^2}{32\pi^2} \epsilon^{\rho\sigma\alpha\beta} (F_b)_{\alpha\beta} \frac{\partial F^b_{\rho\sigma}}{\partial(\partial_0 A^a_\mu)}$$

From the definition of $F^b_{\alpha\beta}$, the derivative appearing in the above equation can be calculated as;

$$\frac{\partial F^b_{\rho\sigma}}{\partial(\partial_0 A^a_\mu)} = \delta^b_a \left[\delta^0_\rho \delta^\mu_\sigma - \delta^0_\sigma \delta^\mu_\rho\right]$$

Using this derivative, we get after some elementary simplification;

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_i)} = -F^{0i}_a - \frac{\theta e^2}{16\pi^2} \epsilon^{0ijk} (F_a)_{jk} = -g^{ij}(E_a)_j - g^{ij}\frac{\theta e^2}{8\pi^2}(B_a)_j = (E_a)_i + \frac{\theta e^2}{8\pi^2}(B_a)_i$$

Using this expression, we get;

$$\mathcal{N} = \int d^3x \frac{1}{ev} \left[E^a_i + \frac{\theta e^2}{8\pi^2} B^a_i\right] D_i \phi^a = \frac{Q}{e} + \frac{\theta e}{8\pi^2} g \tag{6.3}$$

where

$$Q = \frac{1}{v} \int d^3x E^a_i D_i \phi^a, \; g = \frac{1}{v} \int d^3x B^a_i D_i \phi^a$$

are the electrical and magnetic charge operators respectively. Now, the $e^{2\pi i\mathcal{N}} = 1$ condition implies that;

$$\frac{Q}{e} + \frac{\theta e}{8\pi^2} g = n_e \in \mathbb{Z} \Rightarrow Q = en_e - \frac{e\theta n_m}{2\pi}$$

where I used the condition $eg = 4\pi n_m$. Now, we can see that in the presence of the θ term, the magnetic monopole of magnetic charge $n_m = 1$ can obtain the following electrical charge;

$$-\frac{e\theta}{2\pi}$$

which is the same as the QED case except in this case, we also get information on the electrical charge assignments of dyons (n_e, n_m).

7 Olive Montonen and $SL(2,\mathbb{Z})$ duality

Set $\theta = 0$ first. We see that there can be electrically charged states $(n_e, 0)$ and magnetically charged states $(0, n_m)$. Moreover, the mass of an electrically charged state $n_e = 1$ is the mass of W boson i.e. $M_W = ve$ while the mass of the single monopole state is $M_M = vg\,(>> ve$ for weak coupling). We can see that if we want to make a theory in which the roles of electrical and magnetic charges is exchanged, then the following exchanges are required;

$$e \longleftrightarrow g$$

$$M_W \longleftrightarrow M_M$$

Now, the dual theory will be at strong coupling because $e \longleftrightarrow g$ means that small e gets mapped to a large g. Moreover, this proposal is based on the analysis of the classical spectrum as was first done in [7]. However, authors of [7] also point out some obvious problems in the proposal. They are as follows;

1) The BPS solution is based on the assumption of a vanishing $V(\phi)$ but there is no such guarantee that quantum corrections would not introduce non zero corrections in the potential like the (Weinberg Coleman potential).

2) There is an obvious problem of the exact matching of states. The W bosons have spin 1 while the spin of the monopole is zero.

3) There is a big problem because of the fact that the dual theory is at strong coupling. Even if the duality exists, there is no easy way to test the theory.

In [7], authors give additional arguments for the proposal that electric magnetic duality should be an exact duality of the $SO(3)$ Yang Mills Higgs theory despite the problems listed above. As we will see, the first two problems will be solved by incorporating this YMH theory into $N = 4$ supersymmetry. Now, let $\theta \neq 0$ and then,

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{\theta e^2}{32\pi^2}F * F - \frac{1}{2}(D_\mu \phi^a)^2$$

Now, we let

$$A^a_\mu \to \frac{A^a_\mu}{e} \Rightarrow F^a_{\mu\nu} \to \frac{F^a_{\mu\nu}}{e}$$

and then, we get;

$$\mathcal{L} = -\frac{1}{4e^2}F^2 - \frac{\theta}{32\pi^2}F * F - \frac{1}{2}(D_\mu \phi^a)^2$$

with the covariant derivatives changed accordingly. Now, we can see that;

$$-\frac{1}{32\pi^2} Im\left[\frac{\theta}{2\pi} + \frac{4i\pi}{e^2}\right](F + i * F)^2 = -\frac{1}{8e^2}(F^2 - *F^2) - \frac{\theta}{32\pi^2}F * F = -\frac{1}{4e^2}F^2 - \frac{\theta}{32\pi^2}F * F$$

where in the last step, I used the fact that $F^2 = - * F^2$ as can be demonstrated;

$$*F^2 = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = -\frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)F_{\alpha\beta}F^{\rho\sigma} = -F^2$$

So, we can see that the full lagrangian can be written as;

$$\mathcal{L} = -\frac{1}{32\pi^2} Im\,\tau(F + i * F)^2 - \frac{1}{2}(D_\mu\phi^a)^2,\ \tau = \frac{\theta}{2\pi} + \frac{4i\pi}{e^2}$$

I will use a result in order to proceed but the details of the calculation which lead to this result are beyond the scope of this thesis. The result is that the n instanton effects in the theory that we are considering is weighed by $e^{2i\pi n\tau}$ and thus, we can see that the physics is invariant in the change $\tau \to \tau + 1$ which corresponds to $\theta \to \theta + 2\pi$. Moreover, if we set $\theta = 0$, then we can see that the electromagnetic duality corresponds to;

$$e \to g = \frac{4\pi}{e} \Rightarrow \tau = \frac{4i\pi}{e^2} \to -\frac{e^2}{4i\pi} = -\frac{1}{\tau}$$

So, we can identify both of these transformations as special cases of a single general transformation i.e.

$$\tau \to \frac{a\tau + b}{c\tau + d},\ a, b, c, d \in \mathbb{Z},\ ad - cd = 1$$

This transformation is known as $SL(2, \mathbb{Z})$ transformation and the transformation can be identified as follows;

$$\tau \to \tau + 1 \Rightarrow a = b = d = 1,\ c = 0$$
$$\tau \to -\frac{1}{\tau} \Rightarrow a = d = 0,\ c = -b = 1$$

Now, we can also see that;

$$Im(\tau) = \frac{4\pi}{e^2} > 0$$

and thus, we are concerned with the upper τ plane only. Moreover, we can define a fundamental region as;

$$-\frac{1}{2} \leq Re(\tau) \leq \frac{1}{2},\ |\tau| \geq 1$$

This region is important because any τ in the upper half plane can be mapped into this region by an appropriate $SL(2, \mathbb{Z})$ transformation.

7.1 Action of $SL(2, \mathbb{Z})$ on the states

We now turn our attention to the action of the $SL(2, \mathbb{Z})$ group on the states (n_e, n_m). For that, we need the expression for the electrical charge operator Q_e rewritten for reference as follows

$$Q_e = n_e e - \frac{e\theta}{2\pi} n_m \tag{7.1}$$

Now, $\tau \to \tau + 1$ has the following effect;

$$\theta \to \theta + 2\pi \Rightarrow Q \to (n_e - n_m)e - \frac{e\theta}{2\pi} n_m$$

and thus, the state (n_e, n_m) goes to $(n_e - n_m, n_m)$ state and it also corresponds to

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

where I used the fact that for $\tau \to \tau + 1$, $a = b = d = 1$, $c = -1$. Similar procedure can be done for the $\tau \to -\tau^{-1}$ transformation. For this transformation, $\theta = 0$ and we use the following expression for the magnetic charge (alongwith (7.1) for electrical charge)

$$Q_m = g n_m = \frac{4\pi}{e} n_m \tag{7.2}$$

Now, $\tau \to -\tau^{-1}$ exchanges electrical and magnetic charge and thus, we can easily see using (7.1) and (7.2) that it will correspond to $n_e \longleftrightarrow n_m$. In the matrix form, we can write this as;

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

where I used the fact that for $\tau \to -\tau^{-1}$ transformation, we have $a = d = 0$, $c = -b = 1$. So, we it is convincing to conclude that the $SL(2,\mathbb{Z})$ action on the states (n_e, n_m) is;

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

7.2 Revisiting Bogomol'nyi bound

It can be shown that the generalized Bogomol'nyi bound (proved in the appendix)

$$M^2 \geq v^2(Q_e^2 + Q_m^2)$$

can be written entirely in terms of the τ parameter and it is invariant under the $SL(2,\mathbb{Z})$ transformation. We will need the definition of τ and (7.1) – (7.2) for writing it in terms of τ.
We start as;

$$M^2 \geq v^2 \left[\left(n_e - n_m \frac{e\theta}{2\pi} \right)^2 + \left(\frac{4\pi}{e} n_m \right)^2 \right]$$

$$= v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \begin{pmatrix} e^2 & -\frac{e^2\theta}{2\pi} \\ -\frac{e^2\theta}{2\pi} & \left(\frac{4\pi}{e}\right)^2 + \left(\frac{e\theta}{2\pi}\right)^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = 4\pi v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \frac{1}{Im\,\tau} \begin{pmatrix} 1 & -Re\,\tau \\ -Re\,\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

$$\Rightarrow M^2 \geq 4\pi v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \frac{1}{Im\,\tau} \begin{pmatrix} 1 & -Re\,\tau \\ -Re\,\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

8 Coupling to Fermions

If we want to add fermions into this theory, we have to consider the fact that we have a Yang Mills field and a scalar field in the theory already. So, we can write the fermion lagrangian with the appropriate interactions. We assume that fermions are the $'r'$ representation of $SU(2)$ gauge group. The fermion lagrangian is;

$$\mathcal{L} = i\bar{\psi}_n \gamma^\mu (D_\mu \psi)_n - i\bar{\psi}_n T^a_{nm} \phi^a \psi_m \tag{8.1}$$

where T^a_{nm} are the anti-hermitian generators in the representation $'r'$.
Now, we take;

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathcal{I}_2 \\ -\mathcal{I}_2 & 0 \end{pmatrix}, \; \gamma^j = -i \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{8.2}$$

Where $\mathcal{I}_2$ is the 2×2 identity matrix. It can easily be verified that these matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathcal{I}_4$. We can work out the Dirac equation from the lagrangian above (and use $\phi_{nm} = T^a_{nm}\phi^a$) to get;

$$i\gamma^\mu (D_\mu \psi)_n - i\phi_{nm}\psi_m = 0 \tag{8.3}$$

Now, we seek solutions of the form $\psi(t,\mathbf{x}) = e^{-iEt}\psi(\mathbf{x})$ and I also set $\psi(\mathbf{x})_n = \begin{pmatrix}\chi^+(\mathbf{x})_n & \chi^-(\mathbf{x})_n\end{pmatrix}^T$ Using the above mentioned form of solution, the form of the space dependant spinor and the form of the gamma matrices in (8.2), we get;

$$-iE\begin{pmatrix}\chi_n^-\\ -\chi_n^+\end{pmatrix} - \begin{pmatrix}\sigma^j D_j\chi_n^+\\ -\sigma^j D_j\chi_n^-\end{pmatrix} - i\phi_{nm}\begin{pmatrix}\chi_m^+\\ \chi_m^-\end{pmatrix} = 0$$

This gives us two equations. They are as follows;

$$(i\delta_{nm}\sigma^j D_j + \phi_{nm})\chi_m^- = E\chi_n^+ \tag{8.4}$$
$$(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger\chi_m^+ = E\chi_n^- \tag{8.5}$$

Now, if there is some $\psi(\mathbf{x})_n = \begin{pmatrix}\chi^+(\mathbf{x})_n & \chi^-(\mathbf{x})_n\end{pmatrix}^T$ which solves (8.4) – (8.5), with $E = 0$ then that particular solution of fermions would not contribute to the energy of the BPS monopole background and thus, we can have fermionic collective coordinates. These collective coordinates will be coefficients of expansion of the total spinor solution in terms of fermion zero modes (the total solution will also contain non zero modes but the coefficients of expansion of non zero modes won't count as the collective coordinates). In other words, we want to find the kernels of the operators on the left hand side of (8.4) – (8.5). Now, it easy to see that;

$$ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger\right] \subset ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})(i\delta_{ml}\sigma^k D_k + \phi_{ml})^\dagger\right] = \{0\} \tag{8.6}$$

The last equality holds because $(i\delta_{nm}\sigma^j D_j + \phi_{nm})(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger$ is a positive definite operator. Now, we need to find the kernel of $i\delta_{nm}\sigma^j D_j + \phi_{nm}$. A result is needed to find the number of fermion zero modes for different representations $'r'$ in which the fermions live. This result uses the techniques of index theorems and thus, the derivation is beyond the scope of this thesis. The derivative can be found in [8], The final result is;

$$dim\left(ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})\right]\right) - dim\left(\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger\right]\right) = A(r)n_m \tag{8.7}$$

where $A(r)$ is a number that depends on the representation $'r'$ and the ratio of the fermion mass to v where $\lim_{r\to\infty}\phi^a\phi^a = v^2$ and n_m is the magnetic charge. The values of $A(r)$ relevant to our work are as follows;

$$A(\text{fundamental}) = 1 \tag{8.8}$$
$$A(\text{adjoint}) = 2 \tag{8.9}$$

8.1 Fundamental Fermions

Using (8.6) and (8.8), we get to know that there is only one zero mode for the fundamental fermions for a single monopole $n_m = 1$. We will drop the gauge indices for some time now. We have;

$$\psi(\mathbf{x}) = a_0\psi_0(\mathbf{x}) + \text{non zero modes}$$

As mentioned before, a_0 will be our fermionic collective coordinate. The monopole ground states can be built by considering a ground state $|\Omega\rangle$ with $a_0|\Omega\rangle = 0$ and then, we can construct an additional ground state $a_0^\dagger|\Omega\rangle$. So, the states $|\Omega\rangle$ and $a_0^\dagger|\Omega\rangle$ are degenerate.

8.2 Adjoint Fermions

For the fermions in the adjoint representation, we have $A(r) = 2$ for $n_m = 1$ and thus, we have two zero modes. There is an important feature of the monopoles coupled to adjoint fermions. We have to remind ourselves that the monopoles are symmetric in the diagonal subgroup $SU(2)_R \times SU(2)_G$ (Previously I said that it is invariant the diagonal $SO(3)$ subgroup but now, we have added fermions and thus, the $SO(3) = SU(2)/Z_2$ diagonal subgroup is not a symmetry group anymore). The fundamental fermions can be singlets because $\mathbf{2} \times \mathbf{2} = \mathbf{3} + \mathbf{1}$ but this is not the case for the adjoint fermions because $\mathbf{3} \times \mathbf{2} = \mathbf{2} + \mathbf{4}$.
Now, we know that fermions living in **4** representation will be fourfold degenerate. However, we do know that the there are only two fermion zero modes for $n_m = 1$ and thus, fermions can't have spin 3/2 (i.e. they can't be in **4** representation of $SU(2)$). This means that the fermion zero modes will have spins $\pm 1/2$. We can write the fermion solution in terms of fermion zero modes now (with the superscript on zero modes and the additional subscript on the collective coordinates indicating their spins) as;

$$\psi = a_{0,1/2}\psi_0^{1/2} + a_{0,-1/2}\psi_0^{-1/2} + \text{non zero modes}$$

Now, we can make the following degenerate states

State	Spin
$\vert\Omega\rangle$	0
$a^{\dagger}_{0,1/2}\vert\Omega\rangle$	$+\frac{1}{2}$
$a^{\dagger}_{0,-1/2}\vert\Omega\rangle$	$-\frac{1}{2}$
$a^{\dagger}_{0,1/2}a^{\dagger}_{0,-1/2}\vert\Omega\rangle$	0

Table 1: The monopole ground states with adjoint fermions

9 Monopoles in $N = 4$ supersymmetric theories

We need monopoles in $N = 4$ supersymmetry in order to solve two major problems in the Olive Montonen proposal. We don't need to go into the details of the derivation of the $N = 4$ supersymmetry lagrangian. I will use only the result that the $N = 4$ supersymmetric yang mills higgs theory contains only one multiplet (without going into spins higher than 1) with contains one gauge field, six scalars and two Weyl fermions (two Dirac fermions) and all of them are in the adjoint representation since they are in the same multiplet as the gauge field. Now, since there are two Dirac fermions in this theory, the number of fermion zero modes double and thus, we have the following creation operators

$$a^n_{0,\pm 1/2}, \text{ where } n = 1, 2$$

Now, the monopole multiplet is shown in table 2 (we again start with the $|\Omega\rangle$ state and I drop the 0 subscript from the raising operators. Moreover, I replace $\pm 1/2$ with $\pm$ for brevity). We can see that in this multiplet, we have states which have spin 1. Moreover, if we compare the spin content of this multiplet with the $N = 4$ gauge supermultiplet, there is an exact match.
There is another feature of the $N = 4$ theory which is worth mentioning. It is a known fact (I won't go into the details) that the β function of $N = 4$ super Yang Mills theory vanishes and thus, the potential of the $N = 4$ theory won't receive quantum corrections. Therefore, the first two problems

State	Spin
$\lvert\Omega\rangle$	0
$a_{\pm}^{n\dagger}\lvert\Omega\rangle$	$\pm\frac{1}{2}$
$a_{-}^{n\dagger}a_{+}^{m\dagger}\lvert\Omega\rangle$	0
$a_{+}^{1\dagger}a_{+}^{2\dagger}\lvert\Omega\rangle$	1
$a_{-}^{1\dagger}a_{-}^{2\dagger}\lvert\Omega\rangle$	-1
$a_{\mp}^{1\dagger}a_{\mp}^{2\dagger}a_{\pm}^{n\dagger}\lvert\Omega\rangle$	$\mp\frac{1}{2}$
$a_{+}^{1\dagger}a_{+}^{2\dagger}a_{-}^{1\dagger}a_{-}^{2\dagger}\lvert\Omega\rangle$	0

Table 2: The monopole ground states in $N = 4$ supersymmetry

in the Olive Montonen proposal that I mentioned are solved by incorporating 4 dimensional YMH theory in the $N = 4$ supersymmetric gauge theory.

Appendix

In the presence of electrical fields (keeping $A_0^a = 0$ still), we have the following mass for the dyon;

$$M = \frac{1}{2}\int d^3x[E_i^aE_i^a + B_i^aB_i^a + (D_i\phi^a)^2]$$

$$= \frac{1}{2}\int d^3x[(E_i^a - cos\,\alpha D_i\phi^a)^2 + (B_i^a - sin\,\alpha D_i\phi^a)^2] + \int d^3x[cos\,\alpha E_i^aD_i\phi^a + sin\alpha B_i^aD_i\phi^a]$$

$$\geq cos\,\alpha\, Q_e + sin\,\alpha\, Q_m$$

Now, we can make the bound as tight as possible by differentiating the above expression w.r.t α and setting it to zero. This gives;

$$tan\,\alpha = \frac{Q_m}{Q_e},\ sin\,\alpha = \frac{Q_m}{\sqrt{Q_e^2 + Q_m^2}},\ cos\,\alpha = \frac{Q_m}{\sqrt{Q_e^2 + Q_m^2}}$$

This gives us;

$$M \geq v\left[Q_e^2 + Q_m^2\right]^{\frac{1}{2}}$$

This is the generalized Bogomol'nyi bound in terms of (Q_e, Q_m).

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