Review of Seiberg Witten duality

Hassaan Saleem
Email: hsaleem@albany.edu

Supervior: Dr. Bushra Haider
Email: bushrahaider.physics@pu.edu.pk

Abstract

In this work, we start from the superpoincare algebra and review all the concepts needed by the reader to be able to understand the form of the $N = 2$ SYM lagrangian. We will go through the concept of $N = 1$ superfields and the relevant supersymmetric lagrangians that can be made from them before stumbling upon the $N = 2$ SYM lagrangian. In the third chapter, Olive Montonen duality, culminating into $SL(2, \mathbb{Z})$ duality is reviewed. In the fourth chapter, we review Seiberg Witten duality, concluding on the solution of the Higg's expectation value a and it's dual a_D on the $SU(2)$ theory's moduli space (i.e. the complex u plane).

Dedication

Dedicated to my family, friends and all the people (too many to mention here) who have contributed in a positive way to my development as a human being and as a theoretical physicist.

Acknowledgements

I would like to thank Dr. K.S. Narain for very fruitful and scholarly discussions that I had with him in order to develop my understanding required to write this thesis and for the understanding of many other things that I happened to learn from those discussions.

Contents

Chapter 1

Introduction

The main aim of this work is to lay the background material for Seiberg Witten duality and to review the duality itself. This duality is employed to work out the low energy lagrangian for $N = 2$ SYM. The idea of duality however is old and has been employed in solving many problems.

Duality can be defined as the "existence of more than one description (usually two descriptions) for a system". These highly non trivial mathematical equivalences have been very fruitful in the past. For example, the wave particle duality has proven to be very successful in order to explain a large variety of quantum mechanical phenomena. For explaining some phenomena, the particle description is used (for example, to explain the photoelectric effect, light is assumed to consist of photons) and on other occasions, the wave description is used (for example, the explanation of the results of double slit experiments require the wave description). In addition, the harmonic oscillator is a self dual system (explained in the start of chapter 3). An interesting example comes from the theory of ferromagnetism. For the two dimensional Ising model, (which was solved by Lars Onsager in 1943 but published in 1944 [35]) the phase transition temperature was found much earlier by Kramers and Wannier in 1941 using a duality. They used the concept of a dual lattice that led them to relate weak coupling description to the strong coupling description (and they identified the critical temperature by identifying it with the self dual coupling value). So here, duality helps to solve a problem related to a system for which the full solution was not even known [34].

I will touch upon another example (but not go into the details of) in order to show the duality. The Sine Gordon theory for bosons can be shown to be completely equivalent to the Thirring model for fermions (with the interaction term of the form $\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma_{\mu}\psi$) and the remarkable thing is the the strong coupling regime in one theory is equivalent to the weak coupling regime of the other theory. Moreover, the collective excitations of Sine Gordon theory are identified with the fundamental fermion excitations of the Thirring theory. [36, 37]. These points reflect two of the most interesting things because of which the idea of duality is very commonplace in theorists i.e.

1) the possibility of acquiring the capability to study strong coupling regime of one theory by studying the weak coupling regime of another theory (or the same theory sometimes)

2) the mapping of "fundamental excitations" to "collective excitations". This raises the questions like "is it

even meaningful to classify some objects as fundamental?".

A very interesting example of duality (which will be explained at length in chapter 3) is the Olive Montonen duality proposed in 1977 [10]. This duality proposes a dual theory with electric and magnetic degrees of freedom exchanged (therefore, it is also known as electric magnetic duality) and as it turns out, it requires the exchange of strong coupling with weak coupling (thus, this and it's extensions are also known as strong weak duality or S duality). However, in order to realise this theory, it is required that the electric and magnetic degrees of freedom have the same spin (but W bosons had spin 1 and monopoles had spin 0) and that the potential of the theory should not receive quantum corrections (like the Weinberg Coleman potential does for four dimensional scalar QED [38]). For this, it was required that we do put the concerned theory in $N = 4$ supersymmetry as the $N = 4$ gauge multiplet provided the spin 1 state to the monopoles and because the beta function of $N = 4$ theory is zero [19]. This showed the power of supersymmetry in solving important problems.

Later in 1994, N. Seiberg and E. Witten used a duality transformation (which will be discussed in chapter 4) which was used to determine the low energy lagrangian for $N = 2$ SYM theory (i.e. without the matter multiplets) with $SU(2)$ gauge group. This lagrangian is

$$\mathcal{L}_{N=2}(\text{effective}) = \frac{1}{16\pi}\text{Im}\left[\int d^2\theta \mathcal{F}''(\phi)W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\,\bar{\phi}\mathcal{F}'(\phi)\right] \tag{1.1}$$

The low energy theory means that the $SU(2)$ group has spontaneously broken to $U(1)$ and thus, no gauge indices are visible in this lagrangian. Now, the determination of this lagrangian requires us to the find the function $\mathcal{F}(\phi)$ which is known as the prepotential. What Seiberg and Witten did is to solve for the derivative for $\mathcal{F}$ and then, it could be integrated to get $\mathcal{F}$. It should be noted that in their original work [18], Seiberg and Witten used an approach involving elliptic curves to get the solution but in this work, we will use a differential equations approach (which has been used in some works e.g. [20]).

I will start from basics of supersymmetry and develop the relevant concepts to understand the work of Seiberg and Witten in [18]. When some topics arise that are beyond the scope of this work, I will mention so, quote the results and refer the reader to relevant references where possible. I have also provided some appendices for the details that I thought might break the flow of discussion. The reader can consult these appendices if he/she is interested in the details.

Chapter 2

The N=2 SUSY gauge theory

In this thesis, we will talk about $N = 2$ supersymmetric gauge theories (which are based on the vector multiplet). A whole review of $N = 2$ supersymmetric theory won't be given but some details that are crucial for our work will be provided. An interested reader may consult may consult [1] or [2] for a more comprehensive study of supersymmetry. My treatment of supersymmetric theories will follow both of these references to some extent.

2.1 Supersymmetry algebra

We will start from the supersymmetry algebra which is composed of the generators of the Poincare group (P_μ and $M_{\mu\nu}$) and the supersymmetry generators Q^I_α (which are actually Weyl fermions) where I counts the number of supersymmetries and α is the spinor index. The supersymmtery algebra is given as follows [1] (I don't give the commutation relations which correspond to the Poincare group as they would not be needed here but the Poincare algebra has been reviewed in appendix B);

$$[P_\mu, Q^I_\alpha] = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{2.1}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta \tag{2.2}$$

$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \bar{Q}^{I\dot{\beta}} \tag{2.3}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{2.4}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ},\ \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \tag{2.5}$$

where $4(\sigma^{\mu\nu})_\alpha^{\ \beta} = \sigma^\mu_{\alpha\dot{\gamma}} (\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - \mu \leftrightarrow \nu$ and $\sigma_\mu = (1, \sigma_i)$, $\bar{\sigma}_\mu = (1, -\sigma_i)$ (where σ_i are the well known Pauli matrices). Z^{IJ} are known as the central charges of the theory which have to be calculated for a particular theory. For our purpose in the thesis, (i.e. for $N = 2$ super Yang Mills), the central charges were calculated by E. Witten and D. Olive in [4].

2.2 Massless supermultiplets

The relation (2.4) is the most important relation in all of the supersymmetry algebra. It shows that the supersymmetric translations are like a square root of the translations in physical spacetime (remember that P_μ is the generator of the translations in physical spacetime). We now resort to the massless representations in which the momentum four vector can be written as

$$P^\mu = (E, 0, 0, E) \tag{2.6}$$

Using this, (2.4) becomes (for $J = I$);

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2E(\sigma^0_{\alpha\dot{\beta}} + \sigma^3_{\alpha\dot{\beta}}) = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot{\beta}}$$

$$\Rightarrow \{Q^I_1, \bar{Q}^J_{\dot{1}}\} = 0,\ \{Q^I_2, \bar{Q}^I_{\dot{2}}\} = 4E,\ \{Q^I_1, \bar{Q}^I_{\dot{2}}\} = \{Q^I_2, \bar{Q}^I_{\dot{1}}\} = 0 \tag{2.7}$$

(2.7) shows that the central charges vanish for massless representations. (2.7) also implies that Q^I_1 and $\bar{Q}^I_{\dot{1}}$ are trivial on all of the Hilbert space. This can be shown by using the fact that we want all the states in the Hilbert to have positive norm. This requirement is known as the positivity of Hilbert space.

Theorem 2.2.1 (2.7) and the positivity of Hilbert space requires Q^I_1 and $\bar{Q}^I_{\dot{1}}$ to be trivial on Hilbert space (this means that $Q^I_1|\psi\rangle$=0 and $\bar{Q}^I_{\dot{1}}|\psi\rangle = 0$ for all $|\psi\rangle$ in the Hilbert space).
Proof: We any state $|\psi\rangle$ and we consider the following object (which is zero by (2.7));

$$\langle\psi|\{Q^I_1, \bar{Q}^I_{\dot{1}}\}|\psi\rangle$$

This object can be manipulated as follows;

$$0 = \langle\psi|Q^I_1\bar{Q}^I_{\dot{1}}|\psi\rangle + \langle\psi|\bar{Q}^I_{\dot{1}}Q^I_1|\psi\rangle = ||Q^I_1|\psi\rangle||^2 + ||\bar{Q}^I_{\dot{1}}|\psi\rangle||^2$$

Since both of the terms on extreme right hand side are positive definite or zero (due to positivity of Hilbert space), we conclude that Q^I_1 and $\bar{Q}^I_{\dot{1}}$ are trivial on whole of the Hilbert space (QED).

Now, I want to form creation and annihilaton operators using Q^I_2 and $\bar{Q}^I_{\dot{2}}$ operators. This poses the question that which operator should correspond to the creation operator. The answer to this question can be found by analyzing (2.2). We know from our study of Poincare group that $M_{12} = J_3$ (i.e. angular momentum generator in the z direction) and I set $\alpha = 2$ to get;

$$[M_{12}, Q^I_2] = i(\sigma_{12})_2{}^2 Q^I_2 = -\frac{1}{2}Q^I_2 \tag{2.8}$$

where I used calculated $i(\sigma_{12})_2{}^2$ as follows;

$$i(\sigma_{12})_2{}^2 = \frac{i}{4}((\sigma_1)_{2\dot{\gamma}}(\bar{\sigma}_2)^{\dot{\gamma}2} - (\sigma_2)_{2\dot{\gamma}}(\bar{\sigma}_1)^{\dot{\gamma}2}) = \frac{2i^2}{4} = -\frac{1}{2}$$

This shows that we should use Q^I_2 as the annihilation operator and $\bar{Q}^I_{\dot{2}}$ as the creation operator. Now, we define creation and annihilation operator as follows (and we also use (2.7)) to get;

$$a^I = \frac{1}{\sqrt{4E}}Q^I_2,\ (a^\dagger)^I = \frac{1}{\sqrt{4E}}\bar{Q}^I_{\dot{2}} \Rightarrow \{a^I, (a^\dagger)^J\} = 1 \tag{2.9}$$

Now, we can define a vacuum state (known as the Clifford vacuum). We need to remind ourselves that the representations of the Poincare algebra are labelled by the mass of the state and the spin (for massive representations) or by helicity only (for massless representations). This can be understood by considering the fact that Poincare group has $P^{\mu}P_{\mu}$ and $W^{\mu}W_{\mu}$ as it's Casimir operators (where W^{μ} is the Pauli-Lubanski vector). An interested reader might consult [5]. Since we are talking about the massless representations, we deduce that the Clifford vacuum can be specified by the helicity of the state j and thus, it can be written as $|j\rangle$.

Now, we can build the supermultiplet by operating on $|j\rangle$ with the creation operators $(a^{\dagger})^{I}$ while considering the fact that different creation operators anti commute with each other and thus, operating them in a different order would not give a different state. The supermultiplet states and thier multiplicities are given as follows.

States	**Multiplicity**
$\lvert j\rangle$	$\binom{N}{0}$
$(a^{\dagger})^{I}\lvert j\rangle$	$\binom{N}{1}$
$(a^{\dagger})^{I}(a^{\dagger})^{J}\lvert j\rangle$	$\binom{N}{2}$
$\vdots$	$\vdots$
$(a^{\dagger})^{1}...(a^{\dagger})^{N}\lvert j\rangle$	$\binom{N}{N}$
Total	$\binom{N}{0}+\binom{N}{1}+...\binom{N}{N}=2^{N}$

2.3 N=1 multiplets

Now, if we set $N=1$ then there is only one creation operator and let's denote that by $a^{\dagger}$. The two states in the supermultiplet are;

$$|j\rangle,\ a^{\dagger}|j\rangle \tag{2.10}$$

In (2.10), we can set $|j\rangle$ to different possible values to get different supermultiplets. We will need only two multiplets for our work. They are as follows;

$$j=0\rightarrow|0\rangle, a^{\dagger}|0\rangle=\left|\frac{1}{2}\right\rangle\rightarrow\left(0,\frac{1}{2}\right)\underset{\text{CPT}}{\oplus}\left(-\frac{1}{2},0\right)\ \text{(chiral multiplet)} \tag{2.11}$$

$$j=\frac{1}{2}\rightarrow\left|\frac{1}{2}\right\rangle, a^{\dagger}\left|\frac{1}{2}\right\rangle=|1\rangle\rightarrow\left(\frac{1}{2},1\right)\underset{\text{CPT}}{\oplus}\left(-1,-\frac{1}{2}\right)\ \text{(vector/ gauge multiplet)} \tag{2.12}$$

These are the required $N=1$ multiplets in our work. We have to include the CPT conjugate of the mutliplet (which reverses the sign of the helicity) in order to preserve the CPT symmetry.

2.4 N=2 multiplets

If we set $N = 2$ in (2.10) then the length of the mutliplet is $2^2 = 4$. We will have two creation operators i.e. $(a^\dagger)^1$ and $(a^\dagger)^2$ and the states for are Clifford vacuum $|j\rangle$ are;

$$|j\rangle,\ (a^\dagger)^1|j\rangle,\ (a^\dagger)^2|j\rangle,\ (a^\dagger)^1(a^\dagger)^2|j\rangle$$

We are interested in two supermultiplets. They are given as follows;

$$j = 0 \rightarrow |0\rangle\,, (a^\dagger)^1\,|0\rangle = \left|\frac{1}{2}\right\rangle, (a^\dagger)^2|0\rangle = \left|\frac{1}{2}\right\rangle, (a^\dagger)^1(a^\dagger)^2\,|0\rangle = |1\rangle$$

$$\rightarrow \left(0, \frac{1}{2}, \frac{1}{2}, 1\right) \underset{\text{CPT}}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) \text{(vector multiplet)} \tag{2.13}$$

$$j = -\frac{1}{2} \rightarrow \left|-\frac{1}{2}\right\rangle, (a^\dagger)^1\left|-\frac{1}{2}\right\rangle = |0\rangle\,, (a^\dagger)^2\,|0\rangle = \left|\frac{1}{2}\right\rangle, (a^\dagger)^1(a^\dagger)^2\,|0\rangle = |1\rangle$$

$$\rightarrow \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \underset{\text{CPT}}{\oplus} \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \text{(hypermultiplet)} \tag{2.14}$$

2.5 Superspace formalism

The supersymmetry multiplets can be expressed using superspace formalism as well. I will cover the superspace formalism only briefly. For a more detailed treatment of superspace formalism, an interested reader may consult [1] and [2]. The $N = 1$ superspace has coordinates $(x^\mu, \theta, \bar{\theta})$ where x^μ are the ordinary spacetime coordinates and $\theta, \bar{\theta}$ are additional coordinates known as grassman coordinates - or fermionic coordinates - (i.e. they are coordinates which are special kind of numbers known as grassman numbers). θ is a weyl spinor with two components i.e. θ^1 and θ^2. while $\bar{\theta}^1$ and $\bar{\theta}^2$ are their hermitian conjugates. Now, the most interesting property of these grassman coordinates is that they anticommute with each other i.e. if ψ and ξ are any two grassman numbers then;

$$\psi\xi = -\xi\psi \tag{2.15}$$

This leads to the corollary that for any grassman number $\psi^2 = 0$. For grassman coordinates, we can see that $\theta^1\theta^1 = \theta^2\theta^2 = 0$ and $\theta^1\theta^2 = -\theta^2\theta^1$. We will see that there will be other grassman numbers in our work which will not be grassman coordinates e.g. the infinitesimal parameters for SUSY transformations. The property of anticommutativity will be true for all grassman numbers. However, please do note that for any grassman number ψ, $\psi^2 = \psi\psi$ is zero but $\psi\bar{\psi}$ is not necessarily zero.

Now, any function which depends on superspace coordinates can be expressed as a taylor series in these coordiantes. We can assume a ge generic function $F(x, \theta, \bar{\theta})$ and expand it's taylor series but before that, let me introduce the concept of fermionic dot products and the related conventions.

2.6 An aside: Spinor dot products

Suppose that we have two Weyl spinors ψ_α and ξ_β. We can raise the spinor indices α and β by $\epsilon^{\alpha\beta}$ (just like we raise and lower indices by metric tensor in special and general relativity). The reason for using $\epsilon^{\alpha\beta}$ for raising and lowering indices is that the weyl spinors transform by matrices which belong to $SL(2,\mathbb{C})$ group and $\epsilon^{\alpha\beta}$ is the invariant of this group. Recall that x^μ transform in the $SO(1,3)$ group and if $\Lambda^\mu_{\ \nu}$ are the transformation matrices, we had;

$$g_{\alpha\beta} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} g_{\mu\nu},\ \lambda \in SO(1,3) \tag{2.16}$$

where $g_{\mu\nu}$ is the metric tensor. Hence, $g_{\mu\nu}$ is invariant under $SO(1,3)$ trasnformations. Similarly, if $M^\alpha_{\ \beta}$ are matrices in $SL(2,\mathbb{C})$ group, then $\epsilon_{\alpha\beta}$ is invariant because $\epsilon_{\alpha\beta}M^\alpha_{\ \mu}M^\beta_{\ \nu} = \epsilon_{\mu\nu}\, det(M)$ but $M \in SL(2,\mathbb{C})$ and hence, $det(M) = 1$ and hence we have;

$$\epsilon_{\mu\nu} = \epsilon_{\alpha\beta}M^\alpha_{\ \mu}M^\beta_{\ \nu},\ M \in SL(2,\mathbb{C}) \tag{2.17}$$

Now, my convention for $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ is;

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.18}$$

Now, as said before, spinor indices can be raised or lowered by $\epsilon^{\alpha\beta}$. A demonstration is as follows;

$$\psi_1 = \epsilon_{1\alpha}\psi^\alpha = \epsilon_{12}\psi^2 = -\psi^2,\ \bar{\psi}^2 = \epsilon^{\dot{2}\dot{\alpha}}\psi_{\dot{\alpha}} = \epsilon^{\dot{2}\dot{1}}\psi_{\dot{1}} = \psi_{\dot{1}} \tag{2.19}$$

Note the non trivial correspondence between ψ_1 and ψ^2. Now, we have introduced the notion of raising and lowering indices and thus, I can define the dot product between spinors now. The dot products between two undotted spinors ψ and χ is denoted as $\psi\chi$ and defined as;

$$\psi\chi = \psi^\alpha\chi_\alpha,\ \psi\chi = \chi\psi \tag{2.20}$$

in the last equality, I claimed that $\psi\chi = \chi\psi$. This is true and may verified by the reader (please do remember the anti commutativity of the spinors while verifying this). Note the positioning of the spinor indices (they go from top left to bottom right) and this definition is necessary to ensure some desirable properties of the dot product (like it's commutativity). Similarly, the definition of the dot product of two dotted spinors $\bar{\psi}$ and $\bar{\chi}$ can be given as follows;

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}},\ \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi} = (\psi\chi)^\dagger \tag{2.21}$$

Note the different convention for the positioning of the indices. There are some additional identities which are useful in our work with superspace formalism. They are as follows;

$$\xi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\xi \tag{2.22}$$

$$\xi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\xi \tag{2.23}$$

$$(\xi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi} \tag{2.24}$$

$$(\xi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\xi} \tag{2.25}$$

where σ^μ matrices were defined below (2.5) and for clarity, I should tell that $\xi\sigma^\mu\bar{\psi}$ is defined as $\xi^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}$ (following our conventions for dot products). However, note that the dot product conventions do not apply when we are raising or lowering spinor indices.

2.7 Superspace continued

Now, we have introduced the notion of spinor dot products and thus, we can expand an arbitary function of superspace coordinates i.e. $F(x^\mu, \theta, \bar{\theta})$ as follows;

$$F(x^\mu, \theta, \bar{\theta}) = f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta n(x) + \bar{\theta}\bar{\theta}m(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \quad (2.26)$$

Before proceeding, let me quote and motivate some simple mathematical results that concern the superspace formalism. We talk about derivatives first. The derivatives are defined in a straightforward way. The only features which might be novel are the anticommutativity are hermiticity of the derivative. We have;

$$\frac{\delta}{\delta\theta^\alpha}\theta^\beta = \delta_\alpha^\beta, \; \frac{\delta}{\delta\theta^\alpha}\bar{\theta}^{\dot{\alpha}}\theta^\beta = -\bar{\theta}^{\dot{\alpha}}\delta_\alpha^\beta \quad (2.27)$$

From first equality in (2.27), we can easily see that;

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad (2.28)$$

The integrations are defined as follows;

$$\int d\theta^\alpha = 0, \; \int d\theta^{1,2}\theta^{1,2} = 1, \; \int d\theta^\alpha f(\theta^\alpha) = b \text{ where } f(\theta^\alpha) = a + b\theta^\alpha \quad (2.29)$$

To define integrations on the $\theta^1\theta^2$ and $\bar{\theta}^1\bar{\theta}^2$ spaces, we define integration measures $d^2\theta$ and $d^2\bar{\theta}$ as follows;

$$d^2\theta = \frac{1}{2}d\theta^1 d\theta^2, \; d^2\bar{\theta} = \frac{1}{2}d\bar{\theta}^1 d\bar{\theta}^2 \quad (2.30)$$

Moreover, let's calculate $\theta\theta$ and $\bar{\theta}\bar{\theta}$ as follows;

$$\theta\theta = \theta^\alpha\theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = \epsilon_{12}\theta^1\theta^2 + \epsilon_{21}\theta^2\theta^1 = -2\theta^1\theta^2 = 2\theta^2\theta^1 \quad (2.31)$$

$$\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{1}\dot{2}}\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} + \epsilon_{\dot{2}\dot{1}}\bar{\theta}^{\dot{2}}\bar{\theta}^{\dot{1}} = -2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} = 2\bar{\theta}^{\dot{2}}\bar{\theta}^{\dot{1}} \quad (2.32)$$

Using (2.30),(2.31) and (2.32), we can deduce that;

$$\int d^2\theta\,\theta\theta = 1, \; \int d^2\bar{\theta}\,\bar{\theta}\bar{\theta} = 1, \; \int d^2\theta d^2\bar{\theta}\theta\theta\bar{\theta}\bar{\theta} = 1 \quad (2.33)$$

where in the last deduction, we have to use the fact that $\theta\theta$ commutes with $\bar{\theta}\bar{\theta}$.

Now, we can start our work on superspace formalism. We start with the form of supersymmetry generators i.e. $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ (remember that we are working in $N = 1$ susy). Now, we proceed by considering a function $F(x, \theta, \bar{\theta})$ and considering a infinitesimal change in the variables to get $F(x + \delta x, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta})$ and since translation in the fermionic coordinate directions is generated by susy generators, we consider grassman paramteres ϵ and $\bar{\epsilon}$ so that this infinitesimal change can be written as action of translation operator on $F(x, \theta, \bar{\theta})$ as follows;

$$F(x + \delta x, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}) = e^{-i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right)}F(x, \theta, \bar{\theta})e^{i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right)} \quad (2.34)$$

But we can even write $F(x,\theta,\bar{\theta})$ as a translation performed on $F(0,0,0)$. This gives us;

$$F(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta}) = e^{-i\left(\epsilon Q+\bar{\epsilon}\bar{Q}\right)}e^{-i\left(xP+\theta Q+\bar{\theta}\bar{Q}\right)}F(x,\theta,\bar{\theta})e^{i\left(xP+\theta Q+\bar{\theta}\bar{Q}\right)}e^{i\left(\epsilon Q+\bar{\epsilon}\bar{Q}\right)} \tag{2.35}$$

Now, take the last two exponentials in (2.35) and combine them using the Hausdorff Baker Campbell formula which says that;

$$e^A + e^B = e^C, \text{ where } C = A + B + \frac{1}{2}[A,B] + ... \tag{2.36}$$

we will see that we will need the first three terms only. The last two exponentials in (2.35) can thus be combined into a single exponential and the exponent of that exponential is worked out as follows;

$$\text{Exponent } = ixP + i\theta Q + i\bar{\theta}\bar{Q} + i\epsilon Q + i\bar{\epsilon}\bar{Q} - \frac{1}{2}\left[[\theta Q,\epsilon Q] + [\theta Q,\bar{\epsilon}\bar{Q}] + [\bar{\theta}\bar{Q},\epsilon Q] + [\bar{\theta}\bar{Q},\bar{\epsilon}\bar{Q}]\right] \tag{2.37}$$

Now, the four commutators in (2.37) have to be calculated but two of them are zero (i.e. first and last) because they will give the anti commutator of Q with itself or of $\bar{Q}$ with itself and that is proportional to central charges as shown in (2.5) but in $N=1$ susy, the central charges are just zero (which is just another way of saying that Q anti commutes with itself). We calculate the second commutator as follows;

$$[\theta Q,\bar{\epsilon}\bar{Q}] = \theta^\alpha Q_\alpha \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}} - \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\theta^\alpha Q_\alpha = -\theta^\alpha\bar{\epsilon}_{\dot{\beta}}\{Q_\alpha,\bar{Q}^{\dot{\beta}}\} = \theta^\alpha\epsilon^{\dot{\beta}}\{Q_\alpha,\bar{Q}_{\dot{\beta}}\} = 2\theta\sigma^\mu\bar{\epsilon}P_\mu \tag{2.38}$$

where in the second last step, I used $\epsilon^{\dot{\beta}\dot{\rho}}$ to raise the index of $\epsilon_{\dot{\beta}}$ (don't confuse between two ϵ's) and lower the index of $\bar{Q}$. In the last step, I used (1.1). Moreover, by using the anticommutativity of commutator and swapping ϵ with θ, we deduce that;

$$[\bar{\theta}\bar{Q},\epsilon Q] = -2\epsilon\sigma^\mu\bar{\theta}P_\mu \tag{2.39}$$

Using (2.38) and (2.39) in (2.37), we get;

$$\text{Exponent } = i\left(x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}\right)P_\mu + i(\epsilon+\theta)Q + i(\bar{\epsilon}+\bar{\theta})\bar{Q} \tag{2.40}$$

This shows that under susy transformations, the differential changes in the coordinates is as follows;

$$\delta x^\mu = i\left(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}\right) \tag{2.41}$$

$$\delta\theta^\alpha = \epsilon^\alpha \tag{2.42}$$

$$\delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}} \tag{2.43}$$

Note that pure susy transformations also affect x^μ. Now, we do a taylor series expansion of $F(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta})$ and compute δF to get;

$$\begin{aligned}\delta F = F(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta}) - F(x,\theta,\bar{\theta}) &= \partial_\mu F\delta x^\mu + \frac{\partial F}{\partial\theta_\alpha}\epsilon_\alpha + \frac{\partial F}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\epsilon}_\alpha \\ &= i\epsilon^\alpha\left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right)F + i\left(i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\right)\bar{\epsilon}^{\dot{\alpha}}F\end{aligned} \tag{2.44}$$

Compare this with $\delta F = i(\epsilon Q + \bar{\epsilon}\bar{Q})F$, we get the following expressions for Q_α and $\bar{Q}_{\dot{\alpha}}$;

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu,\ \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu, \quad \text{where } \partial_\alpha = \frac{\partial}{\partial\theta^\alpha},\ \bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{2.45}$$

2.8 Chiral super fields

In (2.26), we saw that $F(x,\theta,\bar{\theta})$ has many components and every component corresponds to different particles. Now, if we want $N=1$ representation to be realised in the superspace formalism, then we need to cut short the number of components in F then we need to impose certain conditions on F that will render the rest of the components zero and make F's expansion small. Now, that condition needs to be susy invariant. It means that if the we have an operator D (and it's hermitian conjugate $\bar{D}$) and the condition that we want to impose is that $DF=0$ (or $\bar{D}F=0$), then under a supersymmetric transformation $\delta(DF)=D(\delta F)=0$ (or $\delta(\bar{D}F)=0$) so that the condition $DF=0$ (or $\bar{D}F=0$) is still true. One option is to take the operator D a grassman odd operator (i.e. an operator that anti commutes with grassman numbers) with a spinor index (i.e. D_α)and impose the condition that

$$\{D_\alpha,Q_\beta\}=\{\bar{D}_{\dot{\alpha}},Q_\beta\}=\{D_\alpha,\bar{Q}_{\dot{\beta}}\}=\{\bar{D}_{\dot{\alpha}},Q_{\dot{\beta}}\}=0 \tag{2.46}$$

This will imply that under susy transformations,

$$\delta(D_\alpha F)=i(\epsilon Q+\bar{\epsilon}\bar{Q})(D_\alpha F)=D_\alpha\left(i(\epsilon Q+\bar{\epsilon}\bar{Q})F\right)=0 \tag{2.47}$$

and the same result is true for $\delta(\bar{D}_{\dot{\alpha}}F)$. Now, we can define chiral and anti chiral superfields.
A superfield F is **chiral** if $\bar{D}_\alpha F=0$ and it is **anti chiral** if $D_\alpha F=0$. Before moving on to the analysis of chiral superfields, we have to determine the form of D_α and $\bar{D}_{\dot{\alpha}}$. I won't go into the derivation of the form of these operators but I will just give the result that one gets after some trial and error;

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\Rightarrow\bar{D}_{\dot{\alpha}}=\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}+i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{2.48}$$

This is easily confirmed that these operators anti commute with the susy generators. For example, we have;

$$\{D_\alpha,Q_\beta\}=\{\frac{\partial}{\partial\theta^\alpha}+i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu,-i\frac{\partial}{\partial\theta^\beta}-\sigma^\nu_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_\nu\}=0 \tag{2.49}$$

where the anticommutator vanishes as every term anti commutes individually with every other term. The same thing can be checked for other anticommutators in (2.46).
Now, we do have a form for the operators D_α and $\bar{D}_\alpha$ (also known as covariant derivatives) and thus, we can talk about the chiral super fields. The chiral superfields, as told before obey the condition $\bar{D}_{\dot{\alpha}}=0$. It can be seen easily that they should depend on θ_α as $\bar{D}_{\dot{\alpha}}\theta_\alpha=0$. The dependence on x^μ and $\dot{\theta}_{\dot{\alpha}}$ is more subtle. For investigating this dependence, we introduce a new variable $y^\nu=x^\nu+i\theta^\rho\sigma^\mu_{\beta\dot{\sigma}}\bar{\theta}^{\dot{\sigma}}$ and observe that;

$$\bar{D}_{\dot{\alpha}}y^\nu=\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}+i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\right)\left(x^\nu+i\theta^\rho\sigma^\nu_{\rho\dot{\sigma}}\bar{\theta}^{\dot{\sigma}}\right)=-i\theta^\rho\sigma^\nu_{\rho\dot{\sigma}}\delta^{\dot{\sigma}}_{\dot{\alpha}}+i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\delta^\nu_\mu=0 \tag{2.50}$$

So, a chiral field should be a function of θ_α and y^ν only. The most general expansion of a chiral field (call it ϕ) is;

$$\phi=z(y)+\sqrt{2}\theta\psi(y)-\theta\theta f(y) \tag{2.51}$$

Now, the chiral superfield is like the massless chiral representation of N=1 susy (i.e. a scalar and a spinor). We will see later that $f(y)$ is actually an auxiliary field (i.e. a field with no dynamics or a field without any kinetic term). Now, the expansion can be done in terms of x^μ for future use. We have;

$$z(y^\mu) = z(x^\mu + i\theta^\alpha \sigma^\nu_{\alpha\dot\beta}\bar\theta^{\dot\beta}) + \frac{1}{2}(i\theta\sigma^\mu\bar\theta)(i\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu z(x) = z(x) + i\theta^\alpha\sigma^\nu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\nu z(x) + \frac{1}{2}(i\theta\sigma^\mu\bar\theta)(i\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu z(x)$$

$$= z(x) + i\theta\sigma^\nu\bar\theta\partial_\nu z(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2 z(x) \tag{2.52}$$

$$\sqrt{2}\theta^\alpha\psi_\alpha(y^\mu) = \sqrt{2}\theta^\alpha[\psi_\alpha(x) + i\theta^\rho\sigma^\nu_{\rho\dot\beta}\bar\theta^{\dot\beta}\partial_\nu\psi_\alpha(x)] = \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\partial_\nu\psi\sigma^\nu\bar\theta \tag{2.53}$$

$$\theta\theta f(y) = \theta\theta f(x) + i\theta\theta\theta\sigma^\nu\bar\theta\partial_\nu f(x) = \theta\theta f(x) \tag{2.54}$$

Where in (2.52) and (2.53), I used the following identities

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta,\ (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = -\frac{1}{2}\theta\theta\bar\theta\bar\theta g^{\mu\nu} \tag{2.55}$$

Using (2.52),(2.53) and (2.54) in (2.51), we get;

$$\phi = z(x) + i\theta\sigma^\nu\bar\theta\partial_\nu z(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2 z(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\nu\psi(x)\sigma^\nu\bar\theta - \theta\theta f(x) \tag{2.56}$$

Before we leave this section, I want to say that the a similar procedure can be followed for the anti chiral fields (call them $\bar\phi$) i.e. the fields that satisfy $D_\alpha\bar\phi = 0$ and they depend on $\bar{y^\mu}$ and $\bar\theta^{\dot\alpha}$.

2.9 Vector superfields

Another condition by which we can reduce the number of components in a superfield is to impose the condition that the superfield is real (i.e. it is equal to it's hermitian conjugate). The superfield and it's hermitian conjugte are written as follows;

$$F(x^\mu,\theta,\bar\theta) = f(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta n(x) + \bar\theta\bar\theta m(x) + \theta\sigma^\mu\bar\theta v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta d(x) \tag{2.57}$$

$$F^\dagger(x^\mu,\theta,\bar\theta) = \bar f(x) + \bar\theta\bar\psi(x) + \theta\chi(x) + \theta\theta\bar m(x) + \bar\theta\bar\theta\bar n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) + \theta\theta\bar\theta\bar\rho(x) + \bar\theta\bar\theta\theta\lambda(x) + \theta\theta\bar\theta\bar\theta\bar d(x) \tag{2.58}$$

The condition $F = F^\dagger$ requires that f and d are real and while $\psi = \chi, m = n, \lambda = \rho$. Using this, the superfield becomes

$$F(x^\mu,\theta,\bar\theta) = f(x) + \theta\psi(x) + \bar\theta\bar\psi(x) + \theta\theta n(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \theta\theta\bar\theta\bar\theta d(x) \tag{2.59}$$

with the understanding that f and d are real. Now, we still have a lot of components in this superfield and it doesn't correspond to $N = 1$ vector representation. For countering this problem, we introduce a guage transformation under which the field should be physically equivalent (i.e. the gauge transformed field is physically equivalent to the original one). Now, as we know from our knowledge of QFT, there are two kinds of gauge transformations i.e. abelian and non abelian. We investigate each of them separately.

2.9.1 Abelian gauge

Our knowledge of abelian QFTs (for example, QED) tells us that the appropriate abelian gauge transformation for a vector field v_μ is;

$$v_\mu = v_\mu + \partial_\mu \alpha \tag{2.60}$$

where α is any scalar function. Now, we do see that there is a vector field in F in (2.59) and thus, (2.60) should be the way by which v_μ should transform under an abelian gauge transformation. So, the v_μ term in (2.59) should transform as;

$$\theta\sigma^\mu\bar{\theta}v_\mu = \theta\sigma^\mu\bar{\theta}v_\mu + \theta\sigma^\mu\bar{\theta}\partial_\mu\alpha \tag{2.61}$$

Now, we can see that in the expansion of ϕ in (2.56), there is a term $i\theta\sigma^\nu\bar{\theta}\partial_\nu z(x)$ and thus, the transformation $F \to F + \phi$ might do the job. However, we want F to stay real and thus the transformation should be $F \to F + \phi + \bar{\phi}$. Before deducing the consequences of this transformation, let's write the expression for $\bar{\phi}(x)$ using (2.56) to get;

$$\bar{\phi}(x) = \bar{z}(x) - i\theta\sigma^\mu\bar{\theta}\bar{z}(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\bar{z}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \bar{\theta}\bar{\theta}\bar{f}(x) \tag{2.62}$$

Moreover, we have f and ψ both in F and ϕ, we rename the components of F as $\psi \to \rho, f \to p$. Thus, F becomes;

$$F(x^\mu,\theta,\bar{\theta}) = p(x) + \theta\rho(x) + \bar{\theta}\bar{\rho}(x) + \theta\theta n(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \tag{2.63}$$

Now, the transformation $F \to F + \phi + \bar{\phi}$ implies the following transformation;

$$F \to (p+z+\bar{z}) + \theta(\rho+\sqrt{2}\psi) + \bar{\theta}(\bar{\rho}+\sqrt{2}\bar{\psi}) + \theta\theta(n-f) + \bar{\theta}\bar{\theta}(n-\bar{f}) + \theta\sigma^\mu\bar{\theta}(v_\mu + i\partial_\mu z - i\partial_\mu\bar{z}) + \theta\theta\left(\bar{\lambda} - \frac{i}{\sqrt{2}}\partial_\nu\psi(x)\sigma^\nu\right)\bar{\theta}$$
$$+ \bar{\theta}\bar{\theta}\theta\left(\lambda + \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\psi(x)\right) + \theta\theta\bar{\theta}\bar{\theta}\left(d - \frac{1}{4}\partial^2(z+\bar{z})\right) \tag{2.64}$$

Now, we are free to choose any z, ψ and f. Thus, we choose the following;

$$Re(z) = -\frac{p}{2},\ \psi = -\frac{1}{\sqrt{2}}\rho,\ f = \bar{f} = n \tag{2.65}$$

using these choices, we can guage away p, ρ and n. Using the choice for $Im(z)$, we can gauge away one component of v_μ (just like abelian QFT).This gauge is called **Wess-Zumino gauge**. So, we are left with the following components of the superfield (and since this superfield will correspond to the vector superfield, I will call this superfield V instead of F)

$$V = \theta\sigma^\mu\bar{\theta}v_\mu + \theta\theta\bar{\lambda}\bar{\theta} + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\theta}d$$

We can adopt the convention used normally by the replacements $\lambda \to -i\lambda$, $d \to d/2$. This gives us the following vector superfield;

$$V = \theta\sigma^\mu\bar{\theta}v_\mu + i\theta\theta\bar{\lambda}\bar{\theta} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}d \tag{2.66}$$

This superfield corresponds to a vector superfield as it has a vector and a spinor (we will see that d will be an auxiliary field). An important property of this superfield is that V^3 and higher powers are zero.

2.9.2 Non Abelian gauge

The non abelian generalisation of the gauge transformation will require us to recover the non abelian gauge transformation for a vector field under a gauge group with parameters α_i i.e.

$$v^j_\mu \tau^j \to exp\,(i\alpha_j\tau_j)\left(v^l_\mu\tau^l + \frac{i}{g}\partial_\mu\right)exp\,(-i\alpha_k\tau_k) \tag{2.67}$$

where τ_j are the generators of the gauge group. (for details, see for example [6]). Without going into the details, I can quote the form of the gauge transformation for V that will render the transformation in (2.67) for v_μ. It is given as (see [1])

$$e^V \to e^{i\Omega^\dagger} e^V e^{-i\Omega^\dagger} \tag{2.68}$$

where Ω is a chiral superfield. We can see that at the first order, we recover the abelian gauge transformation by setting $\phi = -i\Omega$ and since we have the same transformation of the first order term in this non abelian gauge as well, we can again choose the choices in (2.65) to p, ρ and n components of V zero. So, even in non abelian gauge, the form of V is still the one given in (2.66) and still, it is the case that V^3 and higher powers vanish.

2.10 Supersymmetric actions

Now, we have almost all the ingredients required to make actions which are invariant in $N = 1$ supersymmetry. The only thing missing is a result which states the susy invariance of certain kind of actions.
The basic result is that any term of the form;

$$\int d^2\theta d^2\bar{\theta} F(x^\mu, \theta, \bar{\theta}) \tag{2.69}$$

(where $F(x^\mu, \theta, \bar{\theta})$ is a general superfield) will be susy invariant. Using (2.44), we can see that;

$$\delta F = \epsilon^\alpha \frac{\partial F}{\partial \theta^\alpha} + \bar{\epsilon}^{\dot{\alpha}} \frac{\partial F}{\partial \bar{\theta}^{\dot{\alpha}}} + i\partial_\mu \left(\theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} F - \epsilon^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} F\right) \tag{2.70}$$

It is from (2.33) that only the terms of the form $\theta\theta\bar{\theta}\bar{\theta}$ (called the **D terms**) will give non zero result for the $d^2\theta d^2\bar{\theta}$ integration. Now, F has only one such term but if we differentiate F w.r.t θ^α or $\bar{\theta}^{\dot{\alpha}}$, then there isn't any such term and thus, the contribution of first two terms in (2.70) in the variation of the integral in (2.69) is zero. The last term in (2.70) is just a total spacetime derivative and thus, it will just give a surface term. So, (2.69) is indeed susy invariant.
There is another result that will be necessary to make super yang mills lagrangians. Suppose that W is a chiral super field. Now, since chiral superfields have a $\theta\theta$ term (called the f term), they will give a non zero answer in the $d^2\theta$ integration. For a chiral field, the second term in (2.70) doesn't appear and the derivative in the last term becomes the derivative w.r.t y^μ. The $d^2\theta$ renders the first term zero again as the θ^α derivative of a chiral superfield doesn't have a $\theta\theta$ term and the second term is again a spacetime derivative. The same thing can be said about the anti chiral fields. The $d^2\bar{\theta}$ integral of an anti chiral superfield will give a susy invariant lagrangian.

2.11 N=1 Super Yang Mills (SYM) lagrangian

For pedagogical purposes, the abelian lagrangian should be built first but I will go to the non abelian theory directly. An interested reader might consult [1] or [2] for the treatment of abelian case.

Now, we have to make a susy lagrangian involving the vector superfield which is given in (2.66) (and the vector fields related by the non Abelian gauge transformation (2.68) are physically equivalent). We can make the required lagrangian by constructing a chiral superfield and then using the fact (as discussed in the last section) that the $d^2\theta$ integral of a chiral superfield is supersymmetric. Now, before making the chiral superfield, I want to point out a fact that will make it easier to see that the chiral field constructed is really chiral. Using the expression for $D_{\dot{\alpha}}$ and $\bar{D}_{\dot{\alpha}}$ from (2.48), we can see that $D^n = \bar{D}^n = 0$ for $n \geq 3$ due to the anticommutivity of the grassman coordinates and the derivatives w.r.t grassman coordinates. Moreover, since there is at least one θ factor in every term of V as well, we can see that $V^n = 0$ for $n \geq 3$ and even if we insert D_α or $\bar{D}_{\dot{\alpha}}$ anywhere in these n factors of V, we still have a vanishing result. For example, for $n = 3$, we have;

$$D_\alpha V^3 = V D_\alpha V^2 = V^2 D_\alpha V = 0 \tag{2.71}$$

It means that any object of the form $\bar{D}^2 P$ (with $\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$) ,where P is any superfield, is chiral as $\bar{D}_{\dot{\alpha}}\bar{D}^2 P = 0$. Using this fact, the chiral superfield that will give the required gauge transformations for v_μ (and for the field tensor $f_{\mu\nu}$ of non abelian yang mills field) is given as follows (it is easily seen to be chiral);

$$W_\alpha = -\frac{1}{4}\bar{D}^2(e^{-V} D_\alpha e^V) \tag{2.72}$$

Now, we can see the gauge transformation of this field as we want the lagrangian that we want to make to be gauge invariant. The transformation of W_α is as follows;

$$W_\alpha \rightarrow -\frac{1}{4}\bar{D}^2\left[e^{i\Omega}e^{-V}e^{-i\Omega^\dagger}D_\alpha\left(e^{i\Omega^\dagger}e^V e^{-i\Omega}\right)\right] = e^{i\Omega}\left(-\frac{1}{4}\bar{D}^2\left(e^{-V}D_\alpha e^V\right)\right)e^{-i\Omega} + e^{i\Omega}\frac{1}{4}\bar{D}^2 D_\alpha e^{-i\Omega} \tag{2.73}$$

where I used the fact that Ω is a chiral field and $\Omega^\dagger$ is an anti-chiral field. Now, the second term in (2.73) vanishes and to prove it, we have to compute the anticommutator of D_α and $\bar{D}_{\dot{\beta}}$ using the expressions from (2.48) as follows;

$$\begin{aligned}\{D_\alpha, \bar{D}_{\dot{\beta}}\} = \{\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\sigma}}\bar{\theta}^{\dot{\sigma}}\partial_\mu, \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} + i\theta^\rho\sigma^\nu_{\rho\dot{\beta}}\partial_\nu\} &= i\sigma^\mu_{\rho\dot{\beta}}\{\frac{\partial}{\partial\theta^\alpha}, \theta^\rho\}\partial_\nu + i\sigma^\mu_{\alpha\dot{\sigma}}\{\bar{\theta}^{\dot{\sigma}}, \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\}\partial_\mu \\ &= i\sigma^\mu_{\rho\dot{\beta}}\delta^\rho_\alpha\partial_\nu + i\sigma^\mu_{\alpha\dot{\sigma}}\delta^{\dot{\sigma}}_{\dot{\beta}}\partial_\mu = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\end{aligned} \tag{2.74}$$

Now,the second term in (2.73) is manipulated as follows;

$$\bar{D}^2 D_\alpha e^{-i\Omega} = -\bar{D}^{\dot{\beta}}\left(D_\alpha, \bar{D}_{\dot{\beta}}e^{-i\Omega} - D_\alpha\bar{D}_{\dot{\beta}}e^{-i\Omega}\right) = \bar{D}^{\dot{\beta}}(2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu e^{-i\Omega}) = 0 \tag{2.75}$$

where I used the fact that Ω is a chiral superfield and the fact that $\bar{D}_{\dot{\beta}}$ and ∂_μ commute. So, (2.73) becomes;

$$W_\alpha \rightarrow e^{i\Omega}\left(-\frac{1}{4}\bar{D}^2\left(e^{-V}D_\alpha e^V\right)\right)e^{-i\Omega} = e^{i\Omega}W_\alpha e^{-i\Omega} \tag{2.76}$$

This already has stated to look like the gauge transformation law for the field tensor $f_{\mu\nu}$ for Yang Mills theory.

Now, the calculation of the term that contributes to the susy action is the $\theta\theta$ term (called the f term). Before doing that, we use the fact that $V^n = 0$ for $n \geq 3$ to get;

$$e^V = 1 + V + \frac{1}{2}V^2 \tag{2.77}$$

Using (2.77), we get;

$$W_\alpha \to -\frac{1}{4}\bar{D}^2\left[\left(1 - V + \frac{1}{2}V^2\right)D_\alpha\left(1 + V + \frac{1}{2}V^2\right)\right] = -\frac{1}{4}\bar{D}^2\left(D_\alpha V + \frac{1}{2}D_\alpha V^2 - VD_\alpha V\right)$$
$$= -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{8}\bar{D}^2[V, D_\alpha V] \tag{2.78}$$

where I dropped the vanishing terms in the second step. Now, we calculate the f term of W_α but before doing that, we change use the $(y^\mu, \theta, \bar{\theta})$ coordinates (where I defined y^μ in the section for chiral superfields i.e. $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$) and then, the form of the covariant derivatives is simplified as follows;

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu,\ \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{2.79}$$

where ∂_μ is $\partial/\partial y^\mu$ now. Moreover, we have to express V (given in (2.66)) in terms of $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ coordinates. We use the anticommutivity of the grassman coordinates to see that only the v_μ term contributes an extra term and all the other terms are zero (because $\theta^n = \bar{\theta}^n = 0$ for $n \geq 3$). We get (now, v_{mu}, λ and d have y^μ as their argument);

$$V = \theta\sigma^\mu\bar{\theta}(v_\mu - i\theta\sigma^\nu\bar{\theta}\partial_\nu v_\mu) + i\theta\theta\bar{\lambda}\bar{\theta} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}d = V = \theta\sigma^\mu\bar{\theta}v_\mu + i\theta\theta\bar{\lambda}\bar{\theta} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(d - i\partial_\mu v^\mu\right) \tag{2.80}$$

where I used the identities

$$2\theta^\alpha\theta^\beta = -\epsilon^{\alpha\beta}\theta\theta,\ 2\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta},\ (\bar{\sigma}^\mu)^{\dot{\sigma}\rho}(\sigma^\nu)_{\rho\dot{\sigma}} = 2g^{\mu\nu}$$

in (2.80). Now, we calculate $D_\alpha V$ as follows;

$$D_\alpha V = \left(\frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\theta}}\partial_\mu\right)V = \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}v_\mu + 2i\theta_\alpha\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\lambda_\alpha + \frac{1}{2}(2\theta_\alpha)\bar{\theta}\bar{\theta}\left(d - i\partial_\mu v^\mu\right) + 2i\sigma^\nu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}(\theta\sigma^\mu\bar{\theta}\partial_\nu v_\mu + i\theta\theta\bar{\theta}\partial_\nu\bar{\lambda})$$
$$= (\sigma^\mu\bar{\theta})_\alpha v_\mu + 2i\theta_\alpha\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\lambda_\alpha + \theta_\alpha\bar{\theta}\bar{\theta}d + 2i(\sigma^{\mu\nu}\theta)_\alpha\bar{\theta}\bar{\theta}\partial_\mu v_\nu + \theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \tag{2.81}$$

where I used the expression for $\sigma^{\mu\nu}$ given under (2.5).Now, applying the $\bar{D}^2$ operator will leave the terms with $\bar{\theta}\bar{\theta}$ factor non zero and the other terms are zero. Before applying this operator, we derive the following fact ;

$$\bar{D}\bar{D}\bar{\theta}\bar{\theta} = \epsilon^{\dot{\alpha}\dot{\rho}}\epsilon_{\dot{\beta}\dot{\sigma}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\rho}}}(\bar{\theta}^{\dot{\sigma}}\bar{\theta}^{\dot{\beta}}) = \epsilon^{\dot{\alpha}\dot{\rho}}\epsilon_{\dot{\beta}\dot{\sigma}}\left(\delta^{\dot{\sigma}}_{\dot{\rho}}\delta^{\dot{\beta}}_{\dot{\alpha}} - \delta^{\dot{\sigma}}_{\dot{\alpha}}\delta^{\dot{\beta}}_{\dot{\rho}}\right) = 2\epsilon^{\dot{\beta}\dot{\rho}}\epsilon_{\dot{\beta}\dot{\rho}} = -2\epsilon^{\dot{\rho}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\rho}} = -2(2) = -4 \tag{2.82}$$

$$\Rightarrow -\frac{1}{4}\bar{D}^2 D_\alpha V = -\frac{1}{4}\bar{D}^2\left(-i\bar{\theta}\bar{\theta}\lambda_\alpha + \theta_\alpha\bar{\theta}\bar{\theta}d + 2i(\sigma^{\mu\nu}\theta)_\alpha\bar{\theta}\bar{\theta}\partial_\mu v_\nu + \theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha\right)$$
$$= -\frac{1}{4}(-4)\left(-i\lambda_\alpha + \theta_\alpha d + 2i(\sigma^{\mu\nu}\theta)_\alpha\partial_\mu v_\nu + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha\right) = -i\lambda_\alpha + \theta_\alpha d + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \tag{2.83}$$

where I used the anti symmetry of $\sigma^{\mu\nu}$ and where $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. Now, the second term in (2.78) is calculated as follows;

$$[V, D_\alpha V] = \bar{\theta}\bar{\theta}(\sigma^{\nu\mu}\theta)_\alpha[v_\mu, v_\nu] + i\theta\theta\bar{\theta}\bar{\theta}\sigma^\mu_{\alpha\dot{\beta}}[v_\mu, \bar{\lambda}^{\dot{\beta}}]$$

$$\Rightarrow \frac{1}{8}\bar{D}^2[V, D_\alpha V] = -\frac{1}{2}(\sigma^{\nu\mu}\theta)_\alpha[v_\mu, v_\nu] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot{\beta}}[v_\mu, \bar{\lambda}^{\dot{\beta}}] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha[v_\mu, v_\nu] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot{\beta}}[v_\mu, \bar{\lambda}^{\dot{\beta}}] \tag{2.84}$$

Using (2.83) and (2.84), and we put these results in (2.78) to get;

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha + \theta_\alpha d + i(\sigma^{\mu\nu}\theta)_\alpha \left(f_{\mu\nu} - \frac{i}{2}[v_\mu, v_\nu]\right) + \theta\theta\left[\sigma^\mu_{\alpha\dot{\beta}}\left(\partial_\mu\bar{\lambda}^{\dot{\beta}} - \frac{i}{2}[v_\mu, \bar{\lambda}^{\dot{\beta}}]\right)\right] \\ &= -i\lambda_\alpha + \theta_\alpha d + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta\left(\sigma^\mu_{\alpha\dot{\beta}} D_\mu\bar{\lambda}^{\dot{\beta}}\right) \end{aligned} \tag{2.85}$$

where, we have;

$$F_{\mu\nu} = f_{\mu\nu} - \frac{i}{2}[v_\mu, v_\nu],\; D_\mu\bar{\lambda}^{\dot{\beta}} = \partial_\mu\bar{\lambda}^{\dot{\beta}} - \frac{i}{2}[v_\mu, \bar{\lambda}^{\dot{\beta}}] \tag{2.86}$$

Now, the term that contributes to the action is the $\theta\theta$ term and using (2.76), we see that the gauge invariant term is $tr(W^\alpha W_\alpha)$. The $\theta\theta$ term is;

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^\mu D_\mu\bar{\lambda} + d^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta}F_{\mu\nu} \tag{2.87}$$

Now, I will use the identity

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \tag{2.88}$$

which can be demonstrated as follows;

and thus, I get;

$$W^\alpha W_\alpha|_{\theta\theta} = \frac{1}{2}F^{\mu\nu}\left(i * F_{\mu\nu} - F_{\mu\nu}\right) - 2i\lambda\sigma^\mu D_\mu\bar{\lambda} + d^2 \tag{2.89}$$

where $*F_{\mu\nu}$ is the dual yang mills field tensor. Now, following the convention (in order to introduce the coupling constant), we let all the components of the vector superfield undergo the following re definitions;

$$v_\mu \to 2gv_\mu,\; d \to 2g\,d,\; \lambda_\alpha \to 2g\lambda_\alpha \tag{2.90}$$

This leads to the following re definitions of $F_{\mu\nu}$ and $D_\mu\bar{\lambda}^{\dot{\beta}}$;

$$F_{\mu\nu} \to 2g\left(\partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]\right) = 2gF^{\text{new}}_{\mu\nu},\; D_\mu\bar{\lambda}^{\dot{\beta}} \to 2g\left(\partial_\mu\bar{\lambda}^{\dot{\beta}} - ig[v_\mu, \bar{\lambda}^{\dot{\beta}}]\right) = 2g\left(D_\mu\bar{\lambda}^{\dot{\beta}}\right)^{\text{new}} \tag{2.91}$$

I will now drop the "new" label from the two redefined objects above and then, (2.89) can be written as;

$$W^\alpha W_\alpha|_{\theta\theta} = 8g^2\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}F^{\mu\nu} * F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}d^2\right) \tag{2.92}$$

Now, it seems that we can simply use (2.92) to calculate the $d^2\theta$ integral of $tr(W^\alpha W_\alpha)$ but there is a problem as we have an imaginary term (i.e. the second term) in (2.92). To counter this problem, we introduce the concept of complex coupling τ as defined below (it will give us the CP violating term which is responsible for Witten effect as we will see in chapter 3). We will multiply the lagrangian with this a multiple of this complex coupling and take the imaginary part to get the final $N = 1$ gauge lagrangian. We first define τ as;

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2} \tag{2.93}$$

where Θ is responsible for the ordinary topological θ term but I have written it as Θ to avoid confusion with the superspace coordinate. The lagrangian is worked out as follows;

$$\begin{aligned} \mathcal{L}_{N=1\,\text{gauge}} &= \text{Im}\left[\frac{\tau}{32\pi}\int d^2\theta\,\text{tr}\left(W^\alpha W_\alpha|_{\theta\theta}\right)\right] = \text{Im}\left[\left(\frac{\Theta g^2}{8\pi^2} + i\right)\text{tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}d^2 + \frac{i}{4}F^{\mu\nu} * F_{\mu\nu}\right)\right] \\ &= \text{tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}d^2 + \frac{\Theta g^2}{32\pi^2}F^{\mu\nu} * F_{\mu\nu}\right) \end{aligned} \tag{2.94}$$

2.12 N=1 gauge matter interaction lagrangian

We can now add matter to the pure Yang Mills field. We know that in among the representations of $N=1$ susy, only the chiral representation is the one which doesn't introduce gravity and which can be put in any representation of the gauge group. So, we take take the superfield corresponding to this representation i.e. the chiral superfield. Now, let the chiral superfield be in any representation of the gauge group (let's call that representation G) and let's call the generators for this representation of gauge group T^a_G, $a=1,2,3,....$ Then the gauge transformation for the chiral superfield is (it is like the gauge transformation of the fermion multiplet in the ordinary Yang Mills theory);

$$\phi^i \to (e^{i\Omega})^i{}_j\phi^j,\ \phi^\dagger_i \to \phi^\dagger_j(e^{-i\Omega^\dagger})^j{}_i, \text{ where } \Omega=\Omega^a T^a_G \text{ and } \Omega \text{ is chiral.} \tag{2.95}$$

Now, the following combination is demonstrated to be gauge invariant (I will use (2.68) in the following demonstration);

$$\phi^\dagger_j(e^V)^j{}_k\phi^k \to \phi^\dagger_j(e^{-i\Omega^\dagger})^j{}_i(e^{i\Omega^\dagger})^i{}_k(e^V)^k{}_l(e^{-i\Omega})^l{}_m(e^{i\Omega})^m{}_n\phi^n = \phi^\dagger_j(e^V)^j{}_k\phi^k \tag{2.96}$$

Now, this combination is used to construct the gauge matter interaction lagrangian. There is something that needs explanation here. If we take a function of ϕ^i only i.e. $W(\phi)$ (or of $\phi^\dagger_i$ only i.e. $\bar{W}(\phi^\dagger)$) called the superpotential, then the $d^2\theta$ integral (or the $d^2\bar{\theta}$ integral) will also give a susy lagrangian. However, our main goal is to get to the $N=2$ susy lagrangian then there can't be a superpotential there as there isn't any superpotential for λ (i.e. the fermion component of the vector superfield) and since in $N=2$ susy, ψ and λ are on same footing (both are in the gauge representation of $N=2$ susy), there can't be a superpotential for ψ and hence for ϕ. So, we won't include these superpotential terms here.

Now, we should know that the combination in (2.96) is not chiral and thus, to make the susy lagrangian, we need to take it's $\theta\theta\bar{\theta}\bar{\theta}$ integral and thus, we need to compute it's $\theta\theta\bar{\theta}\bar{\theta}$ term (i.e. the D term). We first use (2.77) to get;

$$\phi^\dagger e^V\phi = \phi^\dagger\left(1+V+\frac{1}{2}V^2\right)\phi = \phi^\dagger\phi+\phi^\dagger V\phi+\frac{1}{2}\phi^\dagger V^2\phi \tag{2.97}$$

Now, we calculate the D term of the three terms in (2.97) as follows (I will use (2.56) and (2.62) for the expressions of ϕ and $\phi^\dagger$)

$$\phi^\dagger_i\phi^i|_{\bar{\theta}\bar{\theta}\theta\theta} = \left(-\frac{1}{4}(\bar{z}_i\partial^2 z^i+\partial^2\bar{z}_i z^i)+\frac{1}{2}\partial_\mu\bar{z}_i\partial^\mu z^i+\bar{f}_i f^i\right)\bar{\theta}\bar{\theta}\theta\theta - i\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_i\theta\theta\partial_\nu(\psi^i)^\alpha(\sigma^\nu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} + i\bar{\theta}\bar{\theta}\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\psi}^{\dot{\beta}}_i\theta^\rho\psi^i_\rho$$

To bring the last two terms in the familiar $\bar{\theta}\bar{\theta}\theta\theta$ form, we simplify them as follows;

$$-i\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_i\theta\theta\partial_\nu(\psi^i)^\alpha(\sigma^\nu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} = i\theta\theta\left(\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\right)(\bar{\psi}_i)_{\dot{\alpha}}\partial_\nu(\psi^i)^\alpha\sigma^\nu_{\alpha\dot{\beta}} = \left(\frac{i}{2}\partial_\nu\psi^i\sigma^\nu\bar{\psi}_i\right)\bar{\theta}\bar{\theta}\theta\theta \tag{2.98}$$

$$i\bar{\theta}\bar{\theta}\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\psi}^{\dot{\beta}}_i\theta^\rho\psi^i_\rho = i\bar{\theta}\bar{\theta}\left(\frac{i}{2}\epsilon^{\alpha\rho}\theta\theta\right)(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\psi}^{\dot{\beta}}_i\psi^i_\rho = \left(-\frac{i}{2}\psi^i\sigma^\mu\partial_\mu\bar{\psi}\right)\bar{\theta}\bar{\theta}\theta\theta \tag{2.99}$$

So, $\phi^\dagger_i\phi^i|_{\bar{\theta}\bar{\theta}\theta\theta}$ becomes;

$$\phi^\dagger_i\phi^i|_{\bar{\theta}\bar{\theta}\theta\theta} = \left[-\frac{1}{4}\left(\bar{z}_i\partial^2 z^i+\partial^2\bar{z}_i z^i\right)+\frac{1}{2}\partial_\mu\bar{z}_i\partial^\mu z^i+\bar{f}_i f^i+\frac{i}{2}\left(\partial_\mu\psi^i\sigma^\mu\bar{\psi}_i-\psi^i\sigma^\mu\partial_\mu\bar{\psi}\right)\right]\bar{\theta}\bar{\theta}\theta\theta \tag{2.100}$$

Now, the first two terms in (2.100) can be manipulated to give contributions of the form $\partial_\mu \bar{z}_i \partial^\mu z^i$ and total spacetime derivatives (which can be dropped as they won't contribute to the lagrangian). After such a manipulation, (2.100) becomes;

$$\phi_i^\dagger \phi^i|_{\bar{\theta}\bar{\theta}\theta\theta} = \left[\partial_\mu \bar{z}_i \partial^\mu z^i + \bar{f}_i f^i + \frac{i}{2}\left(\partial_\mu \psi^i \sigma^\mu \bar{\psi}_i - \psi^i \sigma^\mu \partial_\mu \bar{\psi}\right)\right]\bar{\theta}\bar{\theta}\theta\theta \tag{2.101}$$

The second term in (2.97) is calculated as follows (using (2.56),(2.62) and (2.66)) and from now, I won't write the gauge indices as they are unnecessary for our work here;

$$\phi^\dagger V\phi|_{\bar{\theta}\bar{\theta}\theta\theta} = \bar{z}\left[(\theta\sigma^\mu\bar{\theta}v_\mu)(i\theta\sigma^\nu\bar{\theta}\partial_\nu z) - i\sqrt{2}\bar{\theta}\bar{\theta}\theta\lambda\theta\psi + \frac{1}{2}\bar{\theta}\bar{\theta}\theta\theta dz\right] - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{z}(\theta\sigma^\mu\bar{\theta}v_\mu z) + \sqrt{2}\bar{\theta}\bar{\psi}\left[\sqrt{2}(\theta\sigma^\mu\bar{\theta}v_\mu)\theta\psi + i\theta\theta\bar{\theta}\bar{\lambda}z\right]$$

The first square term is manipulated as follows

$$-i\bar{z}\left(-\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\right)\left(\frac{1}{2}\epsilon^{\dot{\beta}\dot{\sigma}}\bar{\theta}\bar{\theta}\right)\sigma^\mu_{\alpha\dot{\beta}}\sigma^\nu_{\rho\dot{\sigma}}v_\mu\partial_\nu z + i\sqrt{2}\bar{z}\bar{\theta}\bar{\theta}\left(-\frac{1}{2}\epsilon^{\rho\alpha}\theta\theta\right)\lambda_\rho\psi_\alpha + \frac{1}{2}\bar{z}dz\bar{\theta}\bar{\theta}\theta\theta = \left(\frac{i}{2}\bar{z}v_\mu\partial_\nu z + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi + \frac{1}{2}\bar{z}dz\right)\bar{\theta}\bar{\theta}\theta\theta$$

$$\Rightarrow \bar{z}\left[(\theta\sigma^\mu\bar{\theta}v_\mu)(i\theta\sigma^\nu\bar{\theta}\partial_\nu z) - i\sqrt{2}\bar{\theta}\bar{\theta}\theta\lambda\theta\psi + \frac{1}{2}\bar{\theta}\bar{\theta}\theta\theta dz\right] = \left(\frac{i}{2}\bar{z}v_\mu\partial_\nu z + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi + \frac{1}{2}\bar{z}dz\right)\bar{\theta}\bar{\theta}\theta\theta \tag{2.102}$$

The other terms in $\phi^\dagger V\phi|_{\bar{\theta}\bar{\theta}\theta\theta}$ are calculated as follows;

$$\begin{aligned} &i\theta^\alpha\theta^\rho\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\sigma}}\partial_\mu\bar{z}\sigma^\nu v_\nu z + 2\bar{\theta}_{\dot{\sigma}}\bar{\psi}^{\dot{\sigma}}\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}v_\mu\theta^\rho\psi_\rho + i\sqrt{2}\bar{\theta}_{\dot{\sigma}}\bar{\psi}^{\dot{\sigma}}\theta\theta\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}z \\ &= \left(-\frac{i}{4}(\bar{\sigma}^\mu)^{\dot{\sigma}\rho}\sigma^\nu_{\rho\dot{\sigma}}\partial_\mu\bar{z}v_\nu z + \frac{1}{2}(\bar{\sigma}^\mu)^{\dot{\sigma}\rho}\bar{\psi}_{\dot{\sigma}}v_\mu\psi_\rho + \frac{i}{\sqrt{2}}\psi\lambda z\right)\theta\theta\bar{\theta}\bar{\theta} \\ &= \left(-\frac{i}{2}\partial_\mu\bar{z}\partial^\mu z + \frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi v_\mu + \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z\right)\theta\theta\bar{\theta}\bar{\theta} \end{aligned} \tag{2.103}$$

Using (2.102) and (2.103), we get;

$$\phi^\dagger V\phi|_{\theta\theta\bar{\theta}\bar{\theta}} = \left(\frac{i}{2}\bar{z}v_\mu\partial_\nu z + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi + \frac{1}{2}\bar{z}dz - \frac{i}{2}\partial_\mu\bar{z}\partial^\mu z + \frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi v_\mu + \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z\right)\bar{\theta}\bar{\theta}\theta\theta \tag{2.104}$$

The $d-$term of the third term in (2.97) is fairly simple to calculate as there is only one non zero $d-$term there. It is as follows;

$$\frac{1}{2}\phi^\dagger V^2\phi|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{2}\left[\bar{z}(\theta\sigma^\mu\bar{\theta}v_\mu)(\theta\sigma^\nu\bar{\theta}v_\nu)z\right] = \frac{1}{4}\bar{z}g^{\mu\nu}v_\mu v_\nu z\theta\theta\bar{\theta}\bar{\theta} = \frac{1}{4}\bar{z}v_\mu v^\mu z\theta\theta\bar{\theta}\bar{\theta} \tag{2.105}$$

Using (2.101),(2.104) and (2.105), we get (I will drop the $\bar{\theta}\bar{\theta}\theta\theta$ factor now as it is understood);

$$\begin{aligned} \phi^\dagger e^V\phi|_{\theta\theta\bar{\theta}\bar{\theta}} &= \partial_\mu\bar{z}_i\partial^\mu z^i + \bar{f}_i f^i + \frac{i}{2}\left(\partial_\mu\psi^i\sigma^\mu\bar{\psi}_i - \psi^i\sigma^\mu\partial_\mu\bar{\psi}\right) + \frac{i}{2}\bar{z}v_\mu\partial^\mu z + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi + \frac{1}{2}\bar{z}dz - \frac{i}{2}\partial_\mu\bar{z}v^\mu z \\ &\quad -\frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi v_\mu + \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z + \frac{1}{4}\bar{z}v_\mu v^\mu z \\ &= \psi_\mu\bar{z}D^\mu z + \frac{1}{2}\bar{z}v^\mu\left(D_\mu z + \frac{i}{2}v_\mu z\right) + \frac{i}{2}\left(-\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi\right) + \bar{f}f + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z + \frac{1}{2}\bar{z}dz + \frac{1}{4}\bar{z}v^\mu v_\mu z \\ &= (D_\mu z)^\dagger D_\mu z - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \bar{f}f + \frac{i}{\sqrt{2}}\bar{z}\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z + \frac{1}{2}\bar{z}dz \end{aligned} \tag{2.106}$$

where I have consistently dropped the total spacetime derivatives and used the identity $\psi\sigma^\mu\bar{\xi} = -\xi\bar{\sigma}^\mu\bar{\psi}$. Moreover, the covariant derivative D_μ is defined as follows;

$$D_\mu z = \partial_\mu z - \frac{i}{2}v^j_\mu T^j_G z,\ D_\mu\psi = \partial_\mu\psi - \frac{i}{2}v^j_\mu T^j_G\psi \tag{2.107}$$

Now, we do the same redefinition of the vector superfield as we did in the section of vector superfield i.e. $V \to 2gV (\Rightarrow \lambda \to 2g\lambda,\ v_\mu \to 2gv_\mu,\ d \to 2gd)$. This implied the redefinition of the covariant derivative as follows;

$$D_\mu \to \partial_\mu - \frac{i}{2}(2g)v_\mu = \partial_\mu - ig\, v_\mu = D_\mu^{\text{new}} \tag{2.108}$$

I will drop the "new" label from now on. (2.106) now becomes;

$$\phi^\dagger e^{2gV}\phi|_{\theta\theta\bar{\theta}\bar{\theta}} = (D_\mu z)^\dagger D_\mu z - i\psi\sigma^\mu D_\mu\bar{\psi} + \bar{f}f + ig\sqrt{2}\bar{z}\lambda\psi - ig\sqrt{2}\bar{\psi}\bar{\lambda}z + g\bar{z}dz \tag{2.109}$$

where I used the identity $\psi\sigma^\mu\bar{\xi} = -\xi\bar{\sigma}^\mu\bar{\psi}$ again to write the result in terms of σ^μ instead of $\bar{\sigma}^\mu$ and I dropped the total spacetime derivatives again. Now, the gauge matter lagrangian is extracted by the $\bar{\theta}\bar{\theta}\theta\theta$ integral of $\phi^\dagger e^{2gV}\phi$ to get;

$$\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\bar{\theta}\phi^\dagger e^{2gV}\phi = (D_\mu z)^\dagger D_\mu z - i\psi\sigma^\mu D_\mu\bar{\psi} + \bar{f}f + ig\sqrt{2}\bar{z}\lambda\psi - ig\sqrt{2}\bar{\psi}\bar{\lambda}z + g\bar{z}dz \tag{2.110}$$

Now, I can write the whole $N = 1$ gauge matter lagrangian as follows (i.e. by using (2.94) and (2.110))

$$\begin{aligned}\mathcal{L}_{N=1} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} &= \text{Im}\left[\frac{\tau}{32\pi}\int d^2\theta\, \text{tr}\,(W^\alpha W_\alpha)\right] + \int d^2\bar{\theta}\, d^2\theta\, \phi^\dagger e^{2gV}\phi \\ = \text{tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}d^2 + \frac{\Theta g^2}{32\pi^2}F^{\mu\nu} * F_{\mu\nu}\right) &+ (D_\mu z)^\dagger(D^\mu z) - i\psi\sigma^\mu D_\mu\bar{\psi} + \bar{f}f \\ &+ ig\sqrt{2}\bar{z}\lambda\psi - ig\sqrt{2}\bar{\psi}\bar{\lambda}z + g\bar{z}dz\end{aligned} \tag{2.111}$$

2.13 N=2 Super Yang Mills (SYM) lagrangian

We have to make slight changes to go to $N = 2$ SYM lagrangian. For $N = 2$ lagrangian, we have to put the chiral superfield (in $N = 1$ matter lagrangian) in the adjoint representation. Now, we have to manipulate the terms in the matter part of the lagrangian in (2.111). Before doing that, I will quote some facts about the adjoint representation generators that I will denote by T^a (one of them is actually a convention for the trace that I will choose);

$$[T^a, T^b] = if^{abc}T^c \Rightarrow \text{tr}\left(T^a[T^b, T^c]\right) = if^{abc} \tag{2.112}$$

$$(T^a)_{bc} = -if^{abc} \tag{2.113}$$

$$\text{tr}(T^aT^b) = \delta^{ab} \tag{2.114}$$

Now, we manipulate the terms of the matter part of the lagrangian. We will now put the gauge index on the fields and we have to remind ourselves that gauge index on the chiral superfield will be adjoint index now. as follows;

$$(D_\mu z)^\dagger_a(D^\mu z)^a = (D_\mu z)^\dagger_a(D^\mu z)^b\delta^a_b = \text{tr}\left[(D_\mu z)^\dagger_a T^a(D^\mu z)^b T_b\delta^a_b\right] = \text{tr}\left[(D_\mu z)^\dagger(D^\mu z)\right] \tag{2.115}$$

Similarly,

$$i\psi_a(\sigma^\mu D_\mu\bar{\psi})^a = \text{tr}\left[\psi\sigma^\mu D_\mu\bar{\psi}\right],\ \bar{f}_af^a = \text{tr}\left[\bar{f}f\right] \tag{2.116}$$

Moreover, we have;

$$\bar{z}_a \lambda_b (T^b)^a_{\ c} \psi^c = \bar{z}_a \lambda_b (i f^{ab}_{\ \ c}) \psi^c = \bar{z}_a \lambda_b \psi^c \mathrm{tr}(T^a [T^b, T^c]) = \bar{z}_a \lambda_b \psi^c \mathrm{tr}(T^a T^b T^c - T^a T^c T^b) = \mathrm{tr}\left(z\{\lambda, \psi\}\right) \quad (2.117)$$

Similarly, we have

$$\bar{\psi}\bar{\lambda} z = \mathrm{tr}(\{\bar{\psi}, \bar{\lambda}\} z) \quad (2.118)$$

Lastly, we have;

$$\bar{z}_a d_b (T^b)^a_{\ c} z^c = \bar{z}_a d_b (-i f^{ba}_{\ \ c}) z^c = -\bar{z}_a d_b \mathrm{tr}[T^b [T^a, T^c]] z^c = \mathrm{tr}\left[d[z, \bar{z}]\right] \quad (2.119)$$

where, $z = z_a T^a$, $\psi = \psi_a T^a$, $f = f_a T^a$, $a = 1, 2, 3, ...\mathrm{dim}\ G$ where G is the gauge group.

Now, putting the chiral superfield in the adjoint representation renders the lagrangian to become;

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} = \mathrm{Im}\left[\frac{\tau}{32\pi} \int d^2\theta\, \mathrm{tr}\left(W^\alpha W_\alpha\right)\right] + \int d^2\bar{\theta}\, d^2\theta\, \mathrm{tr}\left(\phi^\dagger e^{2gV} \phi\right)$$

$$= \mathrm{tr}(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} d^2 + \frac{\Theta g^2}{32\pi^2} F^{\mu\nu} * F_{\mu\nu} + (D_\mu z)^\dagger (D^\mu z) - i\psi\sigma^\mu D_\mu \bar{\psi} + \bar{f} f$$

$$+ ig\sqrt{2}\bar{z}\{\lambda, \psi\} - ig\sqrt{2}\{\bar{\psi}, \bar{\lambda}\} z + g d[z, \bar{z}]) \quad (2.120)$$

Now, it is easily seen that this lagrangian in symmetric in the exchange of $\lambda \leftrightarrow \psi$ and it this therefore $N = 2$ supersymmetric. The rigorous way to check $N = 2$ supersymmetry is to check the invariance under the R symmetry that mixes the Q^I_α generators (where $I = 1, 2$) i.e.;

$$Q^I_\alpha \rightarrow U^I_J Q^J_\alpha \quad (2.121)$$

where $U \epsilon U(2)$ (this is determined by the requirement that the norm of the state $Q^I_\alpha |\psi\rangle$ is conserved where $|\psi\rangle$ is just an arbitrary state of the Hilbert space). I won't give the rigorous proof of the $N = 2$ susy of the lagrangian in (2.120) but the invariance of this lagrangian in the exchange $\psi \leftrightarrow \lambda$ strongly suggests the $N = 2$ supersymmetry of this lagrangian. So, we have;

$$\mathcal{L}_{N=2} = \mathrm{Im}\left[\frac{\tau}{32\pi} \int d^2\theta\, \mathrm{tr}\left(W^\alpha W_\alpha\right)\right] + \int d^2\bar{\theta}\, d^2\theta\, \mathrm{tr}\left(\phi^\dagger e^{2gV} \phi\right) \quad (2.122)$$

while the expanded form is given in (2.120).

2.14 An aside: Supersymmetry breaking and low energy lagrangian for $N = 2$ SYM

We can look at the lagrangian in (2.120) and we can see that the field content of the lagrangian is;

$$\text{field content } = v_\mu, \lambda, z, \psi, d, f \quad (2.123)$$

Now, we can see that there are no kinetic terms for d and f fields. It means that they are auxiliary fields and we can simply solve their field equations as follows;

$$\frac{\partial \mathcal{L}}{\partial f^a} = 0 \Rightarrow f^a = 0 \quad (2.124)$$

$$\frac{\partial \mathcal{L}}{\partial d^a} = 0 \Rightarrow d^a + g[z,\bar{z}]^a = 0 \Rightarrow d^a = -g[z.\bar{z}]^a \tag{2.125}$$

where a is the gauge index and $[z,\bar{z}]^a$ is defined by the following manipulation;

$$[z,\bar{z}] = z^b\bar{z}^c[T^b,T^c] = z^b\bar{z}^c(if^{bc}_{\ \ a}T^a) = \left(if^{bca}z_b\bar{z}_c\right)T^a = [z,\bar{z}]^aT_a$$

As a side note, I should mention that since we are working in $N=2$ susy now, we haven't included the superpotential here (as I already discussed it above (2.97)). If we had included the superpotential, then the equation of motion for f^a will include the derivatives of superpotential. Furthermore, from my treatment of the $N=1$ SYM, it is evident that I am assuming the gauge group to be simple (as there is a single coupling constant that governs the lagrangian). We can include $U(1)$ factors in the gauge group and this contributes extra terms in the lagrangian and consequently, in the equation of motion of d^a. These terms are known as **Fayet-Iliopoulos** terms but we won't need them in our work. An interested reader might consult [2] for a detailed discussion on susy breaking.

Now, we should recall what a vacuum state is. A vacuum state is a state which is Lorentz invariant and it is a minimum energy state. This minimum might be global or local where in the case of a local vacuum, tunnelling to the absolute vacuum will occur (an interested reader might consult [13] for this concept). Moreover, we should remind ourselves that what it means for a vacuum state to be invariant in some symmetry of the lagrangian. It means that the vacuum is annihilated by the generators of that symmetry. In the case of supersymmetry, the generators are $Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}$. Firstly, we see that the requirement of the Lorentz invariance renders that the vacuum expectation value (vev) of all the non scalar fields vanish -a result very familiar from the study of ordinary QFT-. It means that $\langle\lambda\rangle = \langle\psi\rangle = \langle v_\mu\rangle = 0$. So, the only nonzero vevs consistent with lorentz invariance are $\langle z\rangle$ and $\langle\bar{z}\rangle$. In other words, the only part of the lagrangian which can have a nonzero vev is the scalar potential (which is a function of z and $z^\dagger$) given below;

$$V(z,\bar{z}) = \bar{f}f + \frac{1}{2}gd^2 = \frac{1}{2}g[z,\bar{z}]_a[z,\bar{z}]^a = \frac{1}{2}g\left([z,\bar{z}]\right)^2 \tag{2.126}$$

where I used the equations of motion for f^a and d^a. Now, it can be seen that the energy of the vacuum (which will be denoted by $|\Omega\rangle$ from now) is equal to the vev of this scalar potential because there is no contribution from the non scalar terms and the derivative terms of the scalar fields (as they will break lorentz invariance of the vacuum).

Now, we will prove that the breaking of susy corresponds to the configurations of $z,\bar{z}$ which render $V(z,\bar{z})$ nonzero. Actually it is very easy to see. We first see that

$$0 \le ||Q^I_\alpha|\Omega\rangle||^2 + ||\bar{Q}^I_{\dot{\alpha}}|\Omega\rangle||^2 = \langle\Omega|\bar{Q}_{\dot{\alpha}}Q^I_\alpha|\Omega\rangle + \langle\Omega|Q^I_\alpha\bar{Q}_{\dot{\alpha}}|\Omega\rangle = \langle\Omega|\{Q^I_\alpha,\bar{Q}_{\dot{\alpha}}\}|\Omega\rangle = 2\sigma^\mu_{\alpha\dot{\alpha}}\langle\Omega|P_\mu|\Omega\rangle \tag{2.127}$$

where I used (2.4) in the last step. Now, we can take the trace (over the $\alpha,\dot{\alpha}$ indices) of extreme right hand side of (2.127) and use the identity $\mathrm{tr}(\sigma^\mu) = \delta^{\mu 0}$ to get;

$$4\langle\Omega|P_0|\Omega\rangle \ge 0 \tag{2.128}$$

Now, using (2.127) and (2.128) that the positive vacuum energy can only happen if at least one of the susy generators doesn't annihilate the vacuum (i.e. if susy is broken). It means that for unbroken susy, it is important that the energy of the vacuum is zero and hence, vev of the scalar potential in (2.126) should vanish.

Before ending this chapter, I need to talk about the effective $N = 2$ SYM lagrangian. We do know that if we are doing any field theory in 4 spacetime dimensions then the terms in the lagrangian (excluding the coupling constant) need to have a mass dimension of 4 in order to be renormalizable i.e. [terms in the lagrangian] ≤ 4 Moreover, we do know from the lagrangian non interacting scalar field z i.e;

$$\mathcal{L} = \frac{1}{2}\partial_\mu z \partial^\mu z \tag{2.129}$$

that $[z] = 1$ (remember that $[\partial_\mu] = 1$). In addition, we do know from the Dirac lagrangian i.e.

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \tag{2.130}$$

that $[\psi] = [\bar{\psi}] = 3/2$. Now, we can see from (2.51) that $[\phi] = [z] = 1$ and $[\phi] = [\theta] + [\psi] \Rightarrow [\theta] = -1/2$. So, we have $[\theta] = [\bar{\psi}] = -1/2$. Moreover, since grassman integration and differentiation has the same effect, the dimension of differentials of grassman coordinates are $[d\theta] = [d\bar{\theta}] = 1/2$. Moreover, please note that using (2.66), we can deduce that $[V] = 0$ as $[v_\mu] = 1$ (it can be seen for example, by considering the interaction term of QED and using the fact that the mass dimension of electron's charge is zero). Moreover, for susy covariant derivatives defined in (2.48) we have $[D_\mu] = [\bar{D}_{\dot{\alpha}}] = [\partial/\partial\theta^\alpha] = 1/2$ and this implies that $[W_\mu] = [\bar{D}^2] + [D_\alpha] = 3/2$. This implies that $[\mathrm{tr}\,(W^\alpha W_\alpha)] = 3$ and in addition, we also conclude that $[\bar{\phi}e^{2gV}\phi] = 2$. These mass dimensions tell us that the $N = 2$ lagrangian given in (2.122) has a mass dimension of 4 and thus, is renormalizable in 4 spacetime dimensions (τ is dimensionless off course).

Now, we can ask the question that if we relax the restriction of renormalizability (i.e. if we ask for an effective theory) then how can the lagrangian in (2.122) change? The answer to this question leads us to a certain class of models known as **non-linear sigma models** where the relation between susy and geometry is also understood. I will not go into the details of such models but an interested reader might consult [1] or [2]. As far as the original literature is concerned on this area, I can cite [14] for $N = 1$ susy and [15], [16], [17] for $N = 2$ susy (For $N = 2$ susy, there are two spaces whose geometry have to be understood and these spaces are known as **Coulomb's branch** and **Higg's branch**. [15] and [16] concern Coulomb's branch while [17] concerns the Higg's branch). I will quote some results from $N = 2$ non linear sigma model which will be useful for us in chapter 4. We can write (2.122) as follows;

$$\mathcal{L}_{N=2} = \mathrm{Im}\left[\frac{\tau}{32\pi}\int d^2\theta\, \delta_{ab}\, W^{\alpha a} W^b_\alpha\right] + \int d^2\theta d^2\bar{\theta}\left(\phi^\dagger e^{2gV}\right)^a \phi_a \tag{2.131}$$

where a and b are gauge indices. Now, the generalisation that we get by allowing this lagrangian to be non-renormalizable is that $\tau\delta_{ab}$ can be replaced by any function of $f_{ab}(z)$ of the scalar component of the chiral superfield and ϕ_a in the second term in the lagrangian can be replaced by a function $K_a(z)$ where;

$$K_a(z) = \frac{\partial K}{\partial z^a} \tag{2.132}$$

where $K(z)$ is a function known as the Kahler potential. The $N = 2$ susy relates the Kahler potential and the $f_{ab}(z)$ function. The relationship is that both of these objects can be written in terms of a single

holomorphic function (i.e. a function which depends on z but doesn't depend on $\bar{z}$) known as **prepotential** and is denotes as $\mathcal{F}(z)$. $K(z,\bar{z})$ and $f_{ab}(z)$ can be written as follows [1];

$$f_{ab}(z) = \frac{g^2}{4\pi i}\mathcal{F}_{ab}(z) \tag{2.133}$$

$$K(z,\bar{z}) = \frac{g^2}{8\pi i}\left(\bar{z}_a \frac{\partial \mathcal{F}(z)}{\partial z_a} - \frac{\partial \bar{\mathcal{F}}(\bar{z})}{\partial \bar{z}_a} z_a\right) = \frac{g^2}{8\pi i}\left(\bar{z}_a F^a(z) - \bar{F}^a(\bar{z}) z_a\right) \tag{2.134}$$

where the superscript a denotes the derivative w.r.t z_α. Using (2.133) and (2.134), we can write (after some simple algebra) the generalized form of (2.131) to get the final result of this section as follows;

$$\mathcal{L}_{N=2}(\text{effective}) = \frac{1}{16\pi}\text{Im}\left[\int d^2\theta\, \mathcal{F}_{ab}(\phi) W^{\alpha a} W^b_\alpha + \int d^2\theta d^2\bar{\theta}\left(\bar{\phi} e^{2gV}\right)^a \mathcal{F}_a(\phi)\right] \tag{2.135}$$

Chapter 3

Magnetic monopoles and Olive-Montonen duality

Note: Most of my treatment of the monopoles follows [7] closely.

3.1 Duality

Duality is the presence of two different perspectives of a single problem or of a single physical system. The main examples of the presence of duality would include the wave particle duality. A physical system can be viewed in the position space wave function (i.e. $\langle x|\psi\rangle$) and it can also be seen in the momentum space wave function (i.e. $\langle p|\psi\rangle$).

Another example for the concept of duality would be seen in the harmonic oscillator with the lagrangian

$$L = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 \tag{3.1}$$

It is easy to see that in the replacement

$$x \to \frac{p}{m\omega}, p \to -m\omega x \tag{3.2}$$

the harmonic oscillator is self dual.

3.2 Electromagnetic duality

3.2.1 Sourceless Maxwell equations

The sourceless Maxwell equations are given as follows (take $c = 1$)

$$\nabla.E = 0 \tag{3.3}$$

$$\nabla.B = 0 \tag{3.4}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{3.5}$$

$$\nabla \times B = \frac{\partial E}{\partial t} \tag{3.6}$$

Now, it is easy to see that in following transformation

$$D : E_i \to B_i, B_i \to -E_i \tag{3.7}$$

(3.3)-(3.6) remain invariant. In fact, we can make this into a continuous transformation like;

$$E_i \to cos\theta E_i + sin\theta B_i; B_i \to -sin\theta E_i + cos\theta B_i \tag{3.8}$$

and it is straightforward to see that (3.3)-(3.6) are still invariant.
We can now introduce the electromagnetic field tensor $F_{\mu\nu}$ with

$$E_i = F_{0i}; B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{3.9}$$

Now, we can translate the D transformation into the terms of $F_{\mu\nu}$ tensor as follows;

$$E_i \to B_i \Rightarrow F_{0i} \to \frac{1}{2}\epsilon_{ijk}F_{jk} = *F_{0i}$$

$$B_i \to -E_i \Rightarrow \frac{1}{2}\epsilon_{ijk}F_{jk} \to -F_{0i} = \frac{1}{2}\epsilon_{ijk} * F_{jk}$$

where

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

Now, we can see that the D transformation can be seen as;

$$F_{\mu\nu} \to *F_{\mu\nu}$$

If there are no monopoles, we can take

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = (\nabla \times A)_i = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j)$$

and (as a reminder, I am using $(+---)$ signature)

$$E_i = F_{0i} = (-\nabla\phi - \frac{\partial A}{\partial t})_i = \partial_0 A_i - \partial_i A_0$$

So, we can conclude that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.10}$$

3.2.2 Electromagnetism in Quantum Mechanics

In Quantum Mechanics, the electromagnetic coupling can be realised by minimum coupling scheme

$$\hat{p}_j \rightarrow -i(\nabla - ieA)_j \tag{3.11}$$

and it is a well known fact that the Schrodinger equation is invariant in the gauge transformation,

$$\psi \rightarrow e^{-ie\chi}; A_i \rightarrow A_i - \frac{i}{e} e^{ie\chi} \nabla e^{-ie\chi} \tag{3.12}$$

where $e^{ie\chi}$ is an element of a $U(1)$ group.

In 1931, Dirac tried to add magnetic monopoles without disturbing the coupling [8]. His work gave a remarkable result which gave a reason for the quantization of electrical charge in the presence of magnetic monopoles. We can understand that work by considering a magnetic monopole at $r = 0$ and since we have a magnetic monopole now, it cant the case that $B_i = (\nabla \times A)_i$ everywhere. However, we can still define A_i in patches and in the region where the patches overlap, they should differ by a gauge transformation. So, far away from the monopole (i.e. say, for $r > r_0$) the magnetic field due to the monopole should be like

$$\mathbf{B} = \frac{g\hat{r}}{4\pi r^2} \tag{3.13}$$

Now,we want some $\mathbf{A}$ to give the above magnetic field by taking its curl. We use the polar coordinate system and in this system, the components of $\mathbf{B}$ for an arbitrary $\mathbf{A}$ are given as;

$$B_r = \frac{1}{rsin\theta}\left[\frac{\partial(sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$B_\theta = -\frac{1}{rsin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial(rsin\theta A_\phi)}{\partial r}\right]$$

$$B_\phi = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

Since we want the ϕ component of $\mathbf{B}$ to be zero, we set A_θ and A_r equal to zero. Moreover, for $\mathbf{B}$ to fall as $1/r^2$, the components should fall as $1/r$ and thus, rA_ϕ should be independent of r. This renders B_θ equal to zero and well. Then for B_r then, we have;

$$B_r = \frac{1}{rsin\theta}\frac{\partial(sin\theta A_\phi)}{\partial\theta} = \frac{g}{4\pi r^2} \Rightarrow A_\phi = \frac{g}{4\pi r}\frac{\xi - cos\theta}{sin\theta}$$

where ξ is arbitrary.

Now, we can choose the vector potential in two patches that cover the S^2 circle i.e. the Northern patch and the Southern patch and call the respective $\mathbf{A}$ as $\mathbf{A}_N$ and $\mathbf{A}_S$. For the northern patch, we choose $\xi = 1$ and for the southern patch, we choose $\xi = -1$. Then, we will have the following vector potentials;

$$\mathbf{A}_N = \frac{g}{4\pi r}\frac{1 - cos\theta}{sin\theta}\hat{\phi} \tag{3.14}$$

$$\mathbf{A}_S = -\frac{g}{4\pi r}\frac{1 + cos\theta}{sin\theta}\hat{\phi} \tag{3.15}$$

The overlap region happens to be $\theta = 90^0$ and the difference of the two potentials in this region is as follows;

$$A_N(\theta = \frac{\pi}{2}) - A_S(\theta = \frac{\pi}{2}) = \frac{g}{2\pi r}\hat{\phi} = -\nabla(-\frac{g}{2\pi}\phi) = -\nabla\chi$$

So, we see that the difference in the overlap zone is indeed a gauge transformation and thus, in the overlap zone, the two potentials are physically equivalent.

Now, the $U(1)$ element (i.e. $e^{ie\chi}$) should be continuous and thus, we have the following condition;

$$e^{ie[\chi(2\pi)-\chi(0)]} = 1 \Rightarrow e[\chi(2\pi) - \chi(0)] = 2\pi n \, (n \in \mathbb{Z}) \tag{3.16}$$

Using the definition of $\chi(\phi)$, we have;

$$g = \chi(0) - \chi(2\pi) \tag{3.17}$$

Using (3.16) and (3.17), we get;

$$eg = 2\pi n \, (n \in \mathbb{Z}) \tag{3.18}$$

(3.18) is known as the Dirac quantization condition.

Some comments are in order. Firstly, this condition implies that the presence of a single monopole in the universe will imply the quantization of electrical charge.

Secondly, this quantization condition implies that $e\chi = 0$ and $e\chi = 2\pi$ actually represent the same $U(1)$ element and thus, the $U(1)$ group is compact. In other words, the presence of monopoles imply the presence of compact $U(1)$ group. Using the contrapositive of this statement we can say that the absence of compact $U(1)$ group will imply the absence of magnetic monopoles. The compact $U(1)$ group will be present in theories where a larger group symmetry has been spontaneously broken to a smaller group containing $U(1)$.

At small distances, the monopoles carry color magnetic charge and the combination of color magnetic charge and ordinary magnetic charge satisfy the Dirac quantization condition. More details are found in [12]

3.3 The t'Hooft, Polyakov monopole

In the Dirac's treatment of the monopole, the interior of the monopole is not studied. The long distance behaviour of the magnetic field is studied and the Dirac's quantization condition is derived. In order to study the interior of a monopole, we study a monopole configuration called t'Hooft, Polyakov monopole. It is nothing but a non abelian Yang Mills Higgs theory in $3 + 1$ dimensions. So, the lagrangian of the theory is as follows;

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) \, , \langle \phi \rangle = v \neq 0 \tag{3.19}$$

The equation of motion for $F^a_{\mu\nu}$ is (also called the Gauss constraint)

$$D_\mu F^{\mu\nu}_a = e \epsilon_{abc} \phi_b D^\nu \phi_c \tag{3.20}$$

Define $F_{\mu\nu}$ as;

$$F_{\mu\nu} = \hat{\phi}^a F^a_{\mu\nu} - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D^\mu \hat{\phi}^c$$

where $\hat{\phi}^a$ is a unit vector in the direction of ϕ^a and for this vector to exist, $\phi^a \neq 0$ anywhere.

Now, using the definition of the Yang Mills field $F^a_{\mu\nu}$, we can express $F_{\mu\nu}$ in terms of A^a_μ. The proof goes as follows.

$$F_{\mu\nu} = \hat{\phi}^a[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu] - \frac{1}{e}\epsilon^{abc}\hat{\phi}^a[\partial_\mu\hat{\phi}^b + e\epsilon^{bde}A^d_\mu\hat{\phi}^e][\partial_\nu\hat{\phi}^c + e\epsilon^{cfg}A^f_\nu\hat{\phi}^g]$$

$$= \partial_\mu(\hat{\phi}^a A^a_\nu) - \partial_\nu(\hat{\phi}^a A^a_\mu) - \frac{1}{e}\epsilon^{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial^\mu\hat{\phi}^c$$

$$+A^d_\mu[\partial_\nu\hat{\phi}^a\delta^{ad} - \partial_\nu\hat{\phi}^c\epsilon^{abc}\epsilon^{bde}\hat{\phi}^a\hat{\phi}^e] - A^f_\nu[\partial_\mu\hat{\phi}^a\delta^{af} + \partial_\mu\hat{\phi}^b\epsilon^{abc}\epsilon^{cfg}\hat{\phi}^a\hat{\phi}^g] + e\epsilon^{abc}\hat{\phi}^a A^b_\mu A^c_\nu - e\epsilon^{abc}\hat{\phi}^a\epsilon^{bde}\epsilon cfg A^d_\mu A^f_\nu\hat{\phi}^e\hat{\phi}^g \quad (3.21)$$

In the equation above, the terms that we want are the first three terms. The terms in the second line vanish. It can be shown as follows.

Take the first bracketed term in (3.3). It is;

$$\partial_\nu\hat{\phi}^a\delta^{ad} - \partial_\nu\hat{\phi}^c\epsilon^{abc}\epsilon^{bde}\hat{\phi}^a\hat{\phi}^e = \partial_\nu\hat{\phi}^d + \partial_\nu\hat{\phi}^c(\delta^{ad}\delta^{ce} - \delta^{ae}\delta^{cd})\hat{\phi}^a\hat{\phi}^e = \partial_\nu\hat{\phi}^d + \hat{\phi}^d\hat{\phi}^e\partial_\nu\hat{\phi}^e - \partial_\nu\hat{\phi}^d = 0$$

Where I used the fact that

$$\hat{\phi}^d\partial_\nu\hat{\phi}^d = \frac{1}{2}\partial_\nu(\hat{\phi}^d\hat{\phi}^d) = \frac{1}{2}\partial_\nu(1) = 0$$

The second bracketed term in the second line of (3.3) can be shown to vanish in a similar way. Now, we need to prove that the final two terms in (3.3) vanish. In other words, we need to show that;

$$\epsilon^{abc}\hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc}\hat{\phi}^a\epsilon^{bde}\epsilon^{cfg}A^d_\mu A^f_\nu\hat{\phi}^e\hat{\phi}^g = 0$$

We can manipulate the last term as

$$\epsilon^{abc}\hat{\phi}^a\epsilon^{bde}\epsilon^{cfg}A^d_\mu A^f_\nu\hat{\phi}^e\hat{\phi}^g = (\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf})\hat{\phi}^a\epsilon^{bde}A^d_\mu A^f_\nu\hat{\phi}^e\hat{\phi}^g = \epsilon^{edb}\hat{\phi}^e A^d_\mu A^b_\nu$$

where I have dropped the vanishing term $\epsilon^{bde}\hat{\phi}^a\hat{\phi}^b\hat{\phi}^e A^d_\mu A^a_\nu$ and used the fact that $\hat{\phi}^e\hat{\phi}^e = 1$.

We can now see that;

$$\epsilon^{abc}\hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc}\hat{\phi}^a\epsilon^{bde}\epsilon^{cfg}A^d_\mu A^f_\nu\hat{\phi}^e\hat{\phi}^g = \epsilon^{abc}\hat{\phi}^a A^b_\mu A^c_\nu - \epsilon^{abc}\hat{\phi}^a A^b_\mu A^c_\nu = 0$$

So, we have shown that;

$$F_{\mu\nu} = \partial_\mu(\hat{\phi}^a A^a_\nu) - \partial_\nu(\hat{\phi}^a A^a_\mu) - \frac{1}{e}\epsilon^{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial^\mu\hat{\phi}^c \quad (3.22)$$

Now, a magnetic field can be defined as;

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \epsilon_{ijk}\partial_j(\hat{\phi}^a A^a_k) - \frac{1}{2e}\epsilon_{ijk}\epsilon^{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c$$

Now, in order to calculate the magnetic flux over a surface, we need to integrate this magnetic field over a sphere. This will receive zero contribution from the first term as it is a curl and the sphere has no boundary. The second term gives the following contribution

$$g = \int_{S^2} dS_i B_i = \frac{1}{2ev^3}\int dS_i\epsilon_{ijk}\epsilon^{abc}\phi^a\partial_j\phi^b\partial_k\phi^c$$

This integral is evaluated as by choosing the form $\phi^a = v\hat{x}^a$ at infinity (where $x^a = r\hat{x}^a$). We simply get;

$$\frac{1}{v^3}\int_{S^2} dS_i\epsilon_{ijk}\epsilon_{abc}\phi^a\partial_j\phi^b\partial_k\phi^c = \int_{S^2} r^2 d\Omega\hat{x}_i\epsilon_{ijk}\epsilon_{abc}\hat{x}_a\partial_j\hat{x}_b\partial_k\hat{x}_c$$

$$= \int_{S^2} d\Omega \hat{x}_i \epsilon_{ijk} \epsilon_{abc} \hat{x}_a [\delta_{bj}\delta_{ck} - 2\delta_{bj}\hat{x}_k\hat{x}_c] \tag{3.23}$$

Where I have dropped the vanishing term and used the identity;

$$\partial_i \hat{x}_j = \frac{1}{r}(\delta_{ij} - \hat{x}_i\hat{x}_j)$$

Now, using the identities;

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj},\ \epsilon_{ijk}\epsilon_{jkl} = 2\delta_{il}$$

the integral in (3.23) becomes;

$$2\int_{S^2} d\Omega \hat{x}_i \hat{x}_a [\delta_{ai} - \delta_{ai} + \hat{x}_a\hat{x}_i] = 8\pi \Rightarrow g = -\frac{4\pi}{e}$$

Now, I can choose another form of ϕ^a by multiplying the azimuthal angle φ by an integer n. In other words, I can choose ϕ^a to be

$$\phi^a(\theta, n\varphi) = v\hat{x}^a(\theta, \varphi)$$

This will simply multiply the above integral n times as we are going through S^2_ϕ n times. So, we get the following condition;

$$eg = 4\pi n,\ n \in \mathbb{Z} \tag{3.24}$$

This is the t'Hooft quantization condition.

3.4 Bogomol'nyi bound and BPS solution

3.4.1 Bogomol'nyi bound

We calculate the energy for the static solution of YMH theory in the gauge $A_0^a = 0$ and we get;

$$E = \int d^3x \left[\frac{1}{2}B_i^a B_i^a + \frac{1}{2}(D_i\phi^a)^2 + V(\phi)\right] = \int d^3x [\frac{1}{2}(B_i^a - D_i\phi^a)^2 + B_i^a D_i\phi^a + V(\phi)] \tag{3.25}$$

Now, using the zeroth component Bianchi identity $\epsilon^{\mu\nu\alpha\beta}D_\nu F^a_{\alpha\beta} = 0$ and using the definition of magnetic field in terms of the $F^a_{\mu\nu}$ tensor i.e. $2B_i^a = \epsilon_{ijk}F^a_{jk}$, we get;

$$\epsilon^{0ijk}D_i F^a_{jk} = D_i B_i^a = 0$$

Moreover, since $B_i^a\phi^a$ is gauge singlet, we have $D_i(B_i^a\phi^a) = \partial_i(B_i^a\phi^a)$ and thus, the second term in the integrand in (3.25) can be written as $\partial_i(B_i^a\phi^a)$ and thus, it is a total derivative which implies that the space integral can be done easily on it by converting it to a surface integral with the integrand $B_i^a\phi^a$. Using the result from the previous section, we can see that this integral is nothing but vg but in order to compensate negative magnetic charge, we write this as vg. Thus, (3.25) becomes;

$$E = vg + \int d^3x [\frac{1}{2}(B_i^a - D_i\phi^a)^2 + V(\phi)] \tag{3.26}$$

We can see that since the integral is positive definite, there will be a lower bound on the energy of the static solution and thus,

$$E \geq vg$$

This is also known as the Bogomol'nyi bound.

Note: I am doing all of this for positive magnetic charge. All of this can be done for negative magnetic charge as well.

3.4.2 BPS solution

If we impose the condition $B_i^a = D_i\phi^a$ (also known as the Bogomol'nyi equation) and set $V(\phi) = 0$ everywhere, then the energy of the field configuration is just vg. There is one thing that needs some mentioning here. Since we are setting $V(\phi) = 0$ everywhere, it seems as if we won't be able to get a nonzero $\langle\phi^a\phi^a\rangle$. However, since we need $\langle\phi^a\phi^a\rangle = v^2$ at spatial infinity, we can impose this as a boundary condition.
If we try to give A_i^a a non zero vev, it will break Lorentz invariance. Moreover, for the theory to have non zero topological charge, ϕ^a should vary at infinity and thus, solutions can't be rotationally invariant. However, we can make the solutions invariant under the diagonal subgroup $SO(3)_s \times SO(3)_g$ where $SO(3)_g$ is the global gauge group. We would not need the detailed form of the BPS solutions in this thesis though.

3.5 Moduli space of the BPS solution

Moduli space is the space of fixed energy and fixed topological charges solutions. The coordinates on the moduli space are called moduli or collective coordinates.
We can we easily see that if $\phi^a(x)$ is a solution, then $\phi^a(x+X)$ is also a solution due to Poincare invariance (where X is a fixed $3-$ vector). So, the moduli space of BPS solution contains $\mathbb{R}^3$ as a product.
We now consider the Gauss constraint (3.20) and expand it to get;

$$D_\mu[\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu] = e\epsilon_{abc}\phi_b[\partial^\nu\phi_c + e\epsilon_{cde}A_d^\nu\phi_e]$$

Setting $\nu = 0$ and working in the $A_a^0 = 0$ gauge, we get;

$$D_i\dot{A}_a^i + e[\phi, \dot{\phi}]_a = 0$$

The linearized form of the Gauss' law for deformations to the BPS solutions $(\delta\phi^a, \delta A_i^a)$ is obtained by letting $A_i^a \to A_i^a + \delta A_i^a, \phi^a \to \phi^a + \delta\phi^a$ and realising that only the deformations are allowed to the time dependant. We get;

$$D_i\delta\dot{A}_a^i + e[\phi, \delta\dot{\phi}]_a = 0$$

$$D_i\delta\dot{A}_a^i + e[\phi, \delta\dot{\phi}]_a = 0 \tag{3.27}$$

Moreover, we can also find the linearized form of the Bogomol'nyi equation by expanding it and then linearizing it as;

$$\epsilon_{ijk}[\partial^j\delta A_a^k + e\epsilon_{abc}A_b^j\delta A_c^k] = \partial_i\delta\phi^a + e\epsilon^{abc}A_i^b\delta\phi^c + e[\delta A_i, \phi]_a$$

This gives us;

$$\epsilon_{ijk}D^j\delta A_a^k = D_i\delta\phi_a + e[\delta A_i, \phi]_a \tag{3.28}$$

The solution to the equations (3.27) and (3.28) are

$$\delta A_0^a = 0,\ \delta\phi^a = 0,\ \delta A_i^a = D_i(\xi(t)\phi^a)$$

where $\xi(t)$ is any arbitrary function of time.
We can see that if $\dot{\xi} = 0$, then the change in A_i^a is a large gauge transformation (i.e. it does not vanish at infinity) and the A_i^a fields which are connected by large gauge transformations are not physically equivalent (unlike small gauge transformations). So, the $U(1)$ element which corresponds to this gauge transformation is $e^{\xi\phi}$ and since the unbroken $U(1)$ group is compact, the coordinate ξ has to be compact and thus, the moduli space of BPS solutions is $\mathbb{R}^3 \times S^1$.
More details on the moduli space of BPS solution are given in [9].

3.6 Witten effect

We can add a theta term in the Yang Mills lagrangian without breaking the gauge invariance. This term is of the form;

$$\mathcal{L}_\theta = -\frac{\theta e^2}{32\pi^2} F_{\mu\nu}^a * F_a^{\mu\nu} \tag{3.29}$$

The prefactor is chosen for the later convenience. We can study the effect of this term for monopoles in the simplest case first (i.e. the QED case with $U(1)$ gauge group).

3.6.1 QED case

For QED, we calculate $F^{\mu\nu} * F_{\mu\nu}$ first to get;

$$F^{\mu\nu} * F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{1}{2}\left[\epsilon^{0ijk}F_{0i}F_{jk} + \epsilon^{i0jk}F_{i0}F_{jk} + \epsilon^{ij0k}F_{ij}F_{0k} + \epsilon^{ijk0}F_{ij}F_{k0}\right]$$
$$= 2\epsilon^{0ijk}F_{0i}F_{jk} = 4\mathbf{E}.\mathbf{B}$$

So, we get;

$$\mathcal{L}_\theta = -\frac{\theta e^2}{8\pi^2}\mathbf{E}.\mathbf{B}$$

For a static monopole, we have;

$$\mathbf{E} = \nabla A_0,\ \mathbf{B} = \nabla \times \mathbf{A} + \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r}$$

This leads to the following lagrangian;

$$L_\theta = \int \mathcal{L}_\theta = \frac{e^2 g\theta}{8\pi^2}\int d^3x A_0 \delta^{(3)}(\mathbf{r}) \tag{3.30}$$

where we used the result;

$$\mathbf{E}.\mathbf{B} = \nabla A_0.\left[\nabla \times A + \frac{g}{4\pi}\frac{\hat{r}}{r}\right] = -\frac{g}{4\pi}A_0\nabla.\left(\frac{\hat{r}}{r}\right) + (\text{surface term}) = -gA_0\delta^{(3)}(\mathbf{r})$$

Now, using the quantization condition $eg = 4\pi$ for a single monopole, (6.2) becomes;

$$L_\theta = -\left[-\frac{e\theta}{2\pi}\int d^3x A_0 \delta^{(3)}(\mathbf{r})\right]$$

This is nothing but the interaction term for a charged particle located at the origin with a background field A_0 with charge;

$$-\frac{e\theta}{2\pi}$$

So, the theta term can give charge to the monopoles.

3.6.2 SU(2) case

Now, we can analyse a similar scenario in the $SU(2)$ theory. For this purpose, we consider the large gauge transformations that act on A_μ. The finite deformation (i.e. without the infinitesimal parameter) can be written as;

$$\delta A^a_\mu = \frac{1}{ev} D_\mu \phi^a$$

where v is written for the dimension to work out correctly (Remember that $[A_\mu] = 1$ and $[D_\mu] = 1$ where I am specifying the mass dimension). Let the generator of this gauge transformation be $\mathcal{N}$. Now, since we are not talking about the fermions right now and since the parameter of this large gauge transformation lives on S^1, we can see that $e^{2\pi i \mathcal{N}} = 1$. Since large gauge transformations act on the field, $\mathcal{N}$ is nothing but the conserved Noetherian current corresponding to the deformations of A^a_μ. It can be easily worked out as follows.

$$\mathcal{N} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_\mu)} \delta A^a_\mu$$

Now, we write the lagrangian as;

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} - \frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} * F^{\mu\nu}_a$$

So, we can calculate the required derivative as;

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_\mu)} = -\frac{1}{2} F^{\alpha\beta}_b \frac{\partial F^b_{\alpha\beta}}{\partial(\partial_0 A^a_\mu)} - \frac{\theta e^2}{32\pi^2} \epsilon^{\rho\sigma\alpha\beta} (F_b)_{\alpha\beta} \frac{\partial F^b_{\rho\sigma}}{\partial(\partial_0 A^a_\mu)}$$

From the definition of $F^b_{\alpha\beta}$, the derivative appearing in the above equation can be calculated as;

$$\frac{\partial F^b_{\rho\sigma}}{\partial(\partial_0 A^a_\mu)} = \delta^b_a \left[\delta^0_\rho \delta^\mu_\sigma - \delta^0_\sigma \delta^\mu_\rho\right]$$

Using this derivative, we get after some elementary simplification;

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_i)} = -F^{0i}_a - \frac{\theta e^2}{16\pi^2} \epsilon^{0ijk} (F_a)_{jk} = -g^{ij} (E_a)_j - g^{ij} \frac{\theta e^2}{8\pi^2} (B_a)_j = (E_a)_i + \frac{\theta e^2}{8\pi^2} (B_a)_i$$

Using this expression, we get;

$$\mathcal{N} = \int d^3x \frac{1}{ev} \left[E^a_i + \frac{\theta e^2}{8\pi^2} B^a_i\right] D_i \phi^a = \frac{Q}{e} + \frac{\theta e}{8\pi^2} g \tag{3.31}$$

where

$$Q = \frac{1}{v} \int d^3x E^a_i D_i \phi^a,\ g = \frac{1}{v} \int d^3x B^a_i D_i \phi^a$$

are the electrical and magnetic charge operators respectively. Now, the $e^{2\pi i \mathcal{N}} = 1$ condition implies that;

$$\frac{Q}{e} + \frac{\theta e}{8\pi^2} g = n_e \in \mathbb{Z} \Rightarrow Q = e n_e - \frac{e\theta n_m}{2\pi}$$

where I used the condition $eg = 4\pi n_m$. Now, we can see that in the presence of the θ term, the magnetic monopole of magnetic charge $n_m = 1$ can obtain the following electrical charge;

$$-\frac{e\theta}{2\pi}$$

which is the same as the QED case except in this case, we also get information on the electrical charge assignments of dyons (n_e, n_m).

3.7 Olive Montonen and $SL(2, \mathbb{Z})$ duality

Set $\theta = 0$ first. We see that there can be electrically charged states $(n_e, 0)$ and magnetically charged states $(0, n_m)$. Moreover, the mass of an electrically charged state $n_e = 1$ is the mass of W boson i.e. $M_W = ve$ while the mass of the single monopole state is $M_M = vg\,(>> ve$ for weak coupling). We can see that if we want to make a theory in which the roles of electrical and magnetic charges is exchanged, then the following exchanges are required;

$$e \longleftrightarrow g$$

$$M_W \longleftrightarrow M_M$$

Now, the dual theory will be at strong coupling because $e \longleftrightarrow g$ means that small e gets mapped to a large g. Moreover, this proposal is based on the analysis of the classical spectrum as was first done in [10]. However, authors of [10] also point out some obvious problems in the proposal. They are as follows;

1) The BPS solution is based on the assumption of a vanishing $V(\phi)$ but there is no such guarantee that quantum corrections would not introduce non zero corrections in the potential like the (Weinberg Coleman potential).

2) There is an obvious problem of the exact matching of states. The W bosons have spin 1 while the spin of the monopole is zero.

3) There is a big problem because of the fact that the dual theory is at strong coupling. Even if the duality exists, there is no easy way to test the theory.

In [10], authors give additional arguments for the proposal that electric magnetic duality should be an exact duality of the $SO(3)$ Yang Mills Higgs theory despite the problems listed above. As we will see, the first two problems will be solved by incorporating this YMH theory into $N = 4$ supersymmetry.

Now, let $\theta \neq 0$ and then,

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{\theta e^2}{32\pi^2} F * F - \frac{1}{2}(D_\mu \phi^a)^2$$

Now, we let

$$A_\mu^a \to \frac{A_\mu^a}{e} \Rightarrow F_{\mu\nu}^a \to \frac{F_{\mu\nu}^a}{e}$$

and then, we get;

$$\mathcal{L} = -\frac{1}{4e^2}F^2 - \frac{\theta}{32\pi^2}F*F - \frac{1}{2}(D_\mu\phi^a)^2$$

with the covariant derivatives changed accordingly. Now, we can see that;

$$-\frac{1}{32\pi^2}Im\left[\frac{\theta}{2\pi}+\frac{4i\pi}{e^2}\right](F+i*F)^2 = -\frac{1}{8e^2}(F^2 - *F^2) - \frac{\theta}{32\pi^2}F*F = -\frac{1}{4e^2}F^2 - \frac{\theta}{32\pi^2}F*F$$

where in the last step, I used the fact that $F^2 = - * F^2$ as can be demonstrated;

$$*F^2 = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = -\frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)F_{\alpha\beta}F^{\rho\sigma} = -F^2$$

So, we can see that the full lagrangian can be written as;

$$\mathcal{L} = -\frac{1}{32\pi^2}Im\,\tau(F+i*F)^2 - \frac{1}{2}(D_\mu\phi^a)^2,\ \tau = \frac{\theta}{2\pi}+\frac{4i\pi}{e^2}$$

I will use a result in order to proceed but the details of the calculation which lead to this result are beyond the scope of this thesis. The result is that the n instanton effects in the theory that we are considering is weighed by $e^{2i\pi n\tau}$ and thus, we can see that the physics is invariant in the change $\tau \to \tau+1$ which corresponds to $\theta \to \theta + 2\pi$. Moreover, if we set $\theta = 0$, then we can see that the electromagnetic duality corresponds to;

$$e \to g = \frac{4\pi}{e} \Rightarrow \tau = \frac{4i\pi}{e^2} \to -\frac{e^2}{4i\pi} = -\frac{1}{\tau}$$

So, we can identify both of these transformations as special cases of a single general transformation i.e.

$$\tau \to \frac{a\tau+b}{c\tau+d},\ a,b,c,d \in \mathbb{Z},\ ad-cd = 1$$

This transformation is known as $SL(2,\mathbb{Z})$ transformation and the transformation can be identified as follows;

$$\tau \to \tau+1 \Rightarrow a=b=d=1,\ c=0$$

$$\tau \to -\frac{1}{\tau} \Rightarrow a=d=0,\ c=-b=1$$

Now, we can also see that;

$$Im(\tau) = \frac{4\pi}{e^2} > 0$$

and thus, we are concerned with the upper τ plane only. Moreover, we can define a fundamental region as;

$$-\frac{1}{2} \leq Re(\tau) \leq \frac{1}{2},\ |\tau| \geq 1$$

This region is important because any τ in the upper half plane can be mapped into this region by an appropriate $SL(2,\mathbb{Z})$ transformation.

3.7.1 Action of $SL(2,\mathbb{Z})$ on the states

We now turn our attention to the action of the $SL(2,\mathbb{Z})$ group on the states (n_e, n_m). For that, we need the expression for the electrical charge operator Q_e rewritten for reference as follows

$$Q_e = n_e e - \frac{e\theta}{2\pi}n_m \tag{3.32}$$

Now, $\tau \to \tau + 1$ has the following effect;

$$\theta \to \theta + 2\pi \Rightarrow Q \to (n_e - n_m)e - \frac{e\theta}{2\pi} n_m$$

and thus, the state (n_e, n_m) goes to $(n_e - n_m, n_m)$ state and it also corresponds to

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

where I used the fact that for $\tau \to \tau + 1$, $a = b = d = 1, c = -1$. Similar procedure can be done for the $\tau \to -\tau^{-1}$ transformation. For this transformation, $\theta = 0$ and we use the following expression for the magnetic charge (alongwith (3.32) for electrical charge)

$$Q_m = g n_m = \frac{4\pi}{e} n_m \tag{3.33}$$

Now, $\tau \to -\tau^{-1}$ exchanges electrical and magnetic charge and thus, we can easily see using (3.32) and (3.33) that it will correspond to $n_e \longleftrightarrow n_m$. In the matrix form, we can write this as;

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

where I used the fact that for $\tau \to -\tau^{-1}$ transformation, we have $a = d = 0, c = -b = 1$. So, we it is convincing to conclude that the $SL(2, \mathbb{Z})$ action on the states (n_e, n_m) is;

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

3.7.2 Revisiting Bogomol'nyi bound

It can be shown that the generalized Bogomol'nyi bound (proved in the appendix)

$$M^2 \geq v^2(Q_e^2 + Q_m^2)$$

can be written entirely in terms of the τ parameter and it is invariant under the $SL(2, \mathbb{Z})$ transformation. We will need the definition of τ and (3.32)− (3.33) for writing it in terms of τ.

We start as;

$$M^2 \geq v^2 \left[\left(n_e - n_m \frac{e\theta}{2\pi} \right)^2 + \left(\frac{4\pi}{e} n_m \right)^2 \right]$$

$$= v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \begin{pmatrix} e^2 & -\frac{e^2\theta}{2\pi} \\ -\frac{e^2\theta}{2\pi} & \left(\frac{4\pi}{e}\right)^2 + \left(\frac{e\theta}{2\pi}\right)^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = 4\pi v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \frac{1}{Im\,\tau} \begin{pmatrix} 1 & -Re\,\tau \\ -Re\,\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

$$\Rightarrow M^2 \geq 4\pi v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \frac{1}{Im\,\tau} \begin{pmatrix} 1 & -Re\,\tau \\ -Re\,\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

3.8 Coupling to Fermions

If we want to add fermions into this theory, we have to consider the fact that we have a Yang Mills field and a scalar field in the theory already. So, we can write the fermion lagrangian with the appropriate interactions. We assume that fermions are the $'r'$ representation of $SU(2)$ gauge group. The fermion lagrangian is;

$$\mathcal{L} = i\bar{\psi}_n \gamma^\mu (D_\mu \psi)_n - i\bar{\psi}_n T^a_{nm} \phi^a \psi_m \tag{3.34}$$

where T^a_{nm} are the anti-hermitian generators in the representation $'r'$.

Now, we take;

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathcal{I}_2 \\ -\mathcal{I}_2 & 0 \end{pmatrix}, \gamma^j = -i \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{3.35}$$

Where $\mathcal{I}_2$ is the 2×2 identity matrix. It can easily be verified that these matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathcal{I}_4$. We can work out the Dirac equation from the lagrangian above (and use $\phi_{nm} = T^a_{nm}\phi^a$) to get;

$$i\gamma^\mu (D_\mu \psi)_n - i\phi_{nm}\psi_m = 0 \tag{3.36}$$

Now, we seek solutions of the form $\psi(t, \mathbf{x}) = e^{-iEt}\psi(\mathbf{x})$ and I also set $\psi(\mathbf{x})_n = \begin{pmatrix} \chi^+(\mathbf{x})_n & \chi^-(\mathbf{x})_n \end{pmatrix}^T$ Using the above mentioned form of solution, the form of the space dependant spinor and the form of the gamma matrices in (3.35), we get;

$$-iE \begin{pmatrix} \chi^-_n \\ -\chi^+_n \end{pmatrix} - \begin{pmatrix} \sigma^j D_j \chi^+_n \\ -\sigma^j D_j \chi^-_n \end{pmatrix} - i\phi_{nm} \begin{pmatrix} \chi^+_m \\ \chi^-_m \end{pmatrix} = 0$$

This gives us two equations. They are as follows;

$$(i\delta_{nm}\sigma^j D_j + \phi_{nm})\chi^-_m = E\chi^+_n \tag{3.37}$$

$$(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger \chi^+_m = E\chi^-_n \tag{3.38}$$

Now, if there is some $\psi(\mathbf{x})_n = \begin{pmatrix} \chi^+(\mathbf{x})_n & \chi^-(\mathbf{x})_n \end{pmatrix}^T$ which solves (3.37)−(3.38), with $E = 0$ then that particular solution of fermions would not contribute to the energy of the BPS monopole background and thus, we can have fermionic collective coordinates. These collective coordinates will be coefficients of expansion of the total spinor solution in terms of fermion zero modes (the total solution will also contain non zero modes but the coefficients of expansion of non zero modes won't count as the collective coordinates). In other words, we want to find the kernels of the operators on the left hand side of (3.37)−(3.38). Now, it easy to see that;

$$ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger\right] \subset ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})(i\delta_{ml}\sigma^k D_k + \phi_{ml})^\dagger\right] = \{0\} \tag{3.39}$$

The last equality holds because $(i\delta_{nm}\sigma^j D_j + \phi_{nm})(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger$ is a positive definite operator.

Now, we need to find the kernel of $i\delta_{nm}\sigma^j D_j + \phi_{nm}$. A result is needed to find the number of fermion zero modes for different representations $'r'$ in which the fermions live. This result uses the techniques of index theorems and thus, the derivation is beyond the scope of this thesis. The derivation can be found in [11], The final result is;

$$dim\left(ker\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})\right]\right) - dim\left(\left[(i\delta_{nm}\sigma^j D_j + \phi_{nm})^\dagger\right]\right) = A(r)n_m \tag{3.40}$$

where $A(r)$ is a number that depends on the representation $'r'$ and the ratio of the fermion mass to v where $\lim_{r\to\infty}\phi^a\phi^a = v^2$ and n_m is the magnetic charge. The values of $A(r)$ relevant to our work are as follows;

$$A(\text{fundamental}) = 1 \tag{3.41}$$

$$A(\text{adjoint}) = 2 \tag{3.42}$$

3.8.1 Fundamental Fermions

Using (3.39) and (3.40), we get to know that there is only one zero mode for the fundamental fermions for a single monopole $n_m = 1$. We will drop the gauge indices for some time now. We have;

$$\psi(\mathbf{x}) = a_0\psi_0(\mathbf{x}) + \text{non zero modes}$$

As mentioned before, a_0 will be our fermionic collective coordinate. The monopole ground states can be built by considering a ground state $|\Omega\rangle$ with $a_0|\Omega\rangle = 0$ and then, we can construct an additional ground state $a_0^\dagger|\Omega\rangle$. So, the states $|\Omega\rangle$ and $a_0^\dagger|\Omega\rangle$ are degenerate.

3.8.2 Adjoint Fermions

For the fermions in the adjoint representation, we have $A(r) = 2$ for $n_m = 1$ and thus, we have two zero modes. There is an important feature of the monopoles coupled to adjoint fermions. We have to remind ourselves that the monopoles are symmetric in the diagonal subgroup $SU(2)_R \times SU(2)_G$ (Previously I said that it is invariant the diagonal $SO(3)$ subgroup but now, we have added fermions and thus, the $SO(3) = SU(2)/Z_2$ diagonal subgroup is not a symmetry group anymore). The fundamental fermions can be singlets because $\mathbf{2}\times\mathbf{2} = \mathbf{3}+\mathbf{1}$ but this is not the case for the adjoint fermions because $\mathbf{3}\times\mathbf{2} = \mathbf{2}+\mathbf{4}$.

Now, we know that fermions living in $\mathbf{4}$ representation will be fourfold degenerate. However, we do know that the there are only two fermion zero modes for $n_m = 1$ and thus, fermions can't have spin 3/2 (i.e. they can't be in $\mathbf{4}$ representation of $SU(2)$). This means that the fermion zero modes will have spins $\pm 1/2$. We can write the fermion solution in terms of fermion zero modes now (with the superscript on zero modes and the additional subscript on the collective coordinates indicating their spins) as;

$$\psi = a_{0,1/2}\psi_0^{1/2} + a_{0,-1/2}\psi_0^{-1/2} + \text{non zero modes}$$

Now, we can make the following degenerate states

State	Spin
$\|\Omega\rangle$	0
$a^\dagger_{0,1/2}\|\Omega\rangle$	$+\frac{1}{2}$
$a^\dagger_{0,-1/2}\|\Omega\rangle$	$-\frac{1}{2}$
$a^\dagger_{0,1/2}a^\dagger_{0,-1/2}\|\Omega\rangle$	0

Table 3.1: The monopole ground states with adjoint fermions

3.9 Monopoles in $N = 4$ supersymmetric theories

We need monopoles in $N = 4$ supersymmetry in order to solve two major problems in the Olive Montonen proposal. We don't need to go into the details of the derivation of the $N = 4$ supersymmetry lagrangian. I will use only the result that the $N = 4$ supersymmetric yang mills higgs theory contains only one multiplet (without going into spins higher than 1) with contains one gauge field, six scalars and two Weyl fermions (two Dirac fermions) and all of them are in the adjoint representation since they are in the same multiplet as the gauge field. Now, since there are two Dirac fermions in this theory, the number of fermion zero modes double and thus, we have the following creation operators

$$a^n_{0,\pm 1/2}, \text{ where } n = 1, 2$$

Now, the monopole multiplet is shown in table 2 (we again start with the $|\Omega\rangle$ state and I drop the 0 subscript from the raising operators. Moreover, I replace $\pm 1/2$ with $\pm$ for brevity). We can see that in this multiplet,

State	Spin
$\|\Omega\rangle$	0
$a^{n\dagger}_{\pm}\|\Omega\rangle$	$\pm\frac{1}{2}$
$a^{n\dagger}_{-}a^{m\dagger}_{+}\|\Omega\rangle$	0
$a^{1\dagger}_{+}a^{2\dagger}_{+}\|\Omega\rangle$	1
$a^{1\dagger}_{-}a^{2\dagger}_{-}\|\Omega\rangle$	-1
$a^{1\dagger}_{\mp}a^{2\dagger}_{\mp}a^{n\dagger}_{\pm}\|\Omega\rangle$	$\mp\frac{1}{2}$
$a^{1\dagger}_{+}a^{2\dagger}_{+}a^{1\dagger}_{-}a^{2\dagger}_{-}\|\Omega\rangle$	0

Table 3.2: The monopole ground states in $N = 4$ supersymmetry

we have states which have spin 1. Moreover, if we compare the spin content of this multiplet with the $N = 4$ gauge supermultiplet, there is an exact match.

There is another feature of the $N = 4$ theory which is worth mentioning. It is a known fact (I won't go into the details) that the β function of $N = 4$ super Yang Mills theory vanishes and thus, the potential of the $N = 4$ theory won't receive quantum corrections. Therefore, the first two problems in the Olive Montonen proposal that I mentioned are solved by incorporating 4 dimensional YMH theory in the $N = 4$ supersymmetric gauge theory.

Chapter 4

Seiberg Witten duality

4.1 Effective N=2 SYM with SU(2) gauge group

The original work of Seiberg and Witten in [3] concerned effective $N = 2$ SYM with $SU(2)$ gauge group. In a later paper [18], they did a similar analysis with hypermultiplets included in $N = 2$ SYM. We will focus on the work done in [3].

We can see from (2.126) that the scalar potential for our concerned lagrangian i.e. (2.135) is proportional to $([z, \bar{z}])^2$ and as we mentioned in the last section of chapter 2, this potential should vanish if we want susy to remain intact (a non zero scalar potential breaks susy as proved in the last section of chapter 2). So, we require $[z, \bar{z}] = 0$. Now, as we are working in $SU(2)$ gauge, the hermitian generators are $\sigma^a/2$ where σ^a are Pauli's spin matrices and thus, using the definition of z below (2.119), we have;

$$z = z^a \frac{\sigma^a}{2} \Rightarrow [z, \bar{z}] = z^a \bar{z}^b [\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = \frac{i}{2} z^a \bar{z}^b \epsilon^{abc} \tag{4.1}$$

where we remind ourselves that the structure constants (i.e. f^{abc}) for $SU(2)$ gauge group is just the Levi civita symbol. Now, since, z^a are complex constants, I can write (I will write j for gauge index now as there might be confusion with the real part of z^a from now on);

$$z = \frac{1}{2}(a_j + ib_j)\sigma_j, \text{ where the summation on } j \text{ is assumed and } a_j, b_j \text{ are real} \tag{4.2}$$

Now, we apply an infinitesimal $SU(2)$ guage transformation on z_j.Remember that z_j is in adjoint representation now and thus, the generators that will be used for the transformation are the adjoint generators for $SU(2)$ as given below;

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.3}$$

Now, an infinitesimal gauge transformation (with parameters ϵ_j) is given as;

$$z_j \to \left(\mathcal{I} + i\epsilon_k \frac{\sigma_k}{2}\right)_{jl} z_l \tag{4.4}$$

$$\Rightarrow \begin{pmatrix} a_1+ib_1 \\ a_2+ib_2 \\ a_3+ib_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i\epsilon_3 & i\epsilon_2 \\ i\epsilon_3 & 1 & -i\epsilon_1 \\ -i\epsilon_2 & i\epsilon_1 & 1 \end{pmatrix} \begin{pmatrix} a_1+ib_1 \\ a_2+ib_2 \\ a_3+ib_3 \end{pmatrix} = \begin{pmatrix} (a_1+b_2\epsilon_3-b_3\epsilon_2)+(b_1+a_3\epsilon_2-a_2\epsilon_3) \\ (a_2+b_3\epsilon_1-b_1\epsilon_3)+(b_2+a_1\epsilon_3-a_3\epsilon_1) \\ (a_3+b_1\epsilon_2-b_2\epsilon_1)+(b_3+a_2\epsilon_1-a_1\epsilon_2) \end{pmatrix} \tag{4.5}$$

Now, let's try to find ϵ_j's such that all the real parts vanish. It will lead us to the following set of equations;

$$\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \tag{4.6}$$

Now, the determinant of the coefficient matrix is;

$$\text{determinant of coefficient matrix } = b_3(-b_1b_2)+b_2(b_1b_3)=0 \tag{4.7}$$

which shows that the determinant is zero and thus, the unique solution for ϵ_j's doesn't exist. So, we can try to make ϵ_j's such that a_1 and a_2 are set to zero. Such a solution is readily found as we have two equations for three unknowns and thus, we can do an $SU(2)$ gauge transformation to set $a_1=a_2=0$.

Now,the form of z and $\bar{z}$ has become;

$$z=\frac{1}{2}a_3\sigma_3+\frac{i}{2}b_j\sigma_j \Rightarrow \bar{z}=\frac{1}{2}a_3\sigma_3-\frac{i}{2}b_j\sigma_j \tag{4.8}$$

Now, we can calculate $[z,\bar{z}]$ to get;

$$[z,\bar{z}]=\frac{1}{4}b_ib_j[\sigma_i,\sigma_j]+\frac{i}{4}b_ia_3[\sigma_i,\sigma_3]+\frac{i}{4}a_3b_i[\sigma_i,\sigma_3] \tag{4.9}$$

=-2i$\epsilon_{ij3}b_ia_3$Now, for this to vanish (to preserve susy), it is easy to see that b_1 and b_2 have to vanish. Thus, the final form of z is;

$$z=\frac{1}{2}(a_3+ib_3)\sigma_3=\frac{1}{2}a\sigma_3 \text{ where } a=a_3+ib_3 \text{ and thus, } a \text{ is complex.} \tag{4.10}$$

Now, we remind ourselves that the finite gauge transformation is $\exp{(i\epsilon_jT_j)}$ where T_j are adjoint generators. We can now set $\epsilon_2=\epsilon_3=0$ and calculate the finite gauge transformation matrix as follows;

$$T_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \Rightarrow (T_1)^2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\Omega \Rightarrow (T^1)^{2n+1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, (T^1)^{2n}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, n\in\mathbb{Z}^+$$

$$\exp{(i\epsilon_1T_1)}=1-\Omega+\Omega\cos\epsilon_1+iT_1\sin\epsilon_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon_1 & \sin\epsilon_1 \\ 0 & -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix} \tag{4.11}$$

which is just the rotation matrix around $1-$ axis (which is not surprising). Now, we can act this finite gauge transformation on z which is in it's final form i.e. (4.10) to get;

$$z_j \Rightarrow \exp{(i\epsilon_1T_1)}_{jk}z_k=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon_1 & \sin\epsilon_1 \\ 0 & -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}=\begin{pmatrix} 0 \\ a\sin\epsilon_1 \\ a\cos\epsilon_1 \end{pmatrix} \tag{4.12}$$

Now, we can set $\epsilon_1 = \pi$ and then, we can see that effectively, $a \to -a$. This means that a is itself not gauge invariant but a^2 is. So, $\langle z \rangle$ itself is not gauge invariant but $\langle \mathrm{tr} z^2 \rangle$ is. We will call this gauge invariant thing u from now on and thus, we have the following definitions;

$$u = \langle \mathrm{tr} z^2 \rangle,\ \langle z \rangle = \frac{1}{2} a \sigma_3 \tag{4.13}$$

Now, as discussed in chapter 3, the moduli space refers to the space of vaccua (i.e. space of lowest energy configurations) and we will see that Seiberg Witten duality concerns the moduli space of the effective N=2 lagrangian. So, before proceeding, it should be made clear that u (and not $\langle z \rangle$) labels the gauge inequivalent vaccua as a and $-a$ are not gauge inequivalent. Moreover, since u is a complex number, the moduli space of our concerned lagrangian is just a complex plane (called the u plane).

Now, we can see from (4.13) that z has a non zero vev only for the third component (i.e. for z^3 component) and thus, we can expand the scalar field around this vev (let's call it η^j) as follows;

$$z^j = \eta^j + \langle z^j \rangle = \eta^j + \frac{1}{2} a \delta^{3j} \tag{4.14}$$

Using this expansion, we can see that the expression for $D_\mu z^j$ becomes;

$$D_\mu z^j = \partial_\mu z^j - ig[v_\mu, z]^j = \partial_\mu z^j + g v_\mu^k \eta^l f^{ljk} + \frac{ag}{2} v_\mu^k f^{3jk} \tag{4.15}$$

The last term will generate masses for v_μ in the term $(D_\mu z)^\dagger D_\mu z$ but we can see that due to anti symmetry of f^{ijk}, there won't be any mass term for v_μ^3 and thus, it will remain massless. Moreover, due to the non zero vev of z, $SU(2)$ gauge symmetry is broken to $U(1)$ (a well known result). Now, in this abelian gauge transformation, we do know that the vector field transforms as $v_\mu \to v_\mu + \partial_\mu \alpha$ where α is any arbitrary differentiable function. As discussed in subsection 2.9.1 The corresponding gauge transformation for the vector superfield V is $V \to V + \Omega + \bar{\Omega}$ where ϕ is a chiral superfield. This means that the gauge invariant quantity is $\bar{\phi}\phi$ instead of $\bar{\phi} e^{2gV} \phi$ as can be shown easily;

$$\bar{\phi}\phi = \bar{\phi}^3 \phi_3 \to \bar{\phi}^3 e^{-i\alpha} e^{-i\alpha} \phi_3 = \bar{\phi}^3 \phi_3 \tag{4.16}$$

where I did the $U(1)$ gauge transformation with $3-$ component only. In other words, only 1 in the expansion of e^{2gV} (given in (2.77)) contributes. A small calculation similar to (4.15) shows that the vev of z also gives mass to $\lambda^{1,2}$ and $\psi^{1,2}$ (Remember that supersymmetry requires that the mass of z^j, ψ^j, v_μ^j and λ^j are the same. It can be seen by considering the fact that P^2 commutes with all the susy generators where $P^2 = P_\mu P^\mu$ is the contraction of the momentum generator with itself). In summary, only λ^3, v_μ^3 and ψ^3 are massless and the other fields are massive. Thus, in an effective wilsonian action, the heavier degrees of freedom are integrated out and thus, only the massless fields should participate. Thus, only λ^3, v_μ^3 and ψ^3 fields contribute to the effective action and thus, the gauge index can be dropped. Hence, (2.135) becomes;

$$\mathcal{L}_{N=2}(\text{effective}) = \frac{1}{16\pi} \mathrm{Im} \left[\int d^2\theta \mathcal{F}''(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\, \bar{\phi} \mathcal{F}'_a \right] \tag{4.17}$$

where the primes on $\mathcal{F}$ denote the derivative w.r.t z^3 which will be written just as z as the gauge index is unnecessary now.

4.2 Role of $\mathcal{F}''(z)$ as a metric

Let's analyse (4.17) as bit further. Let's start by the term dependent on the chiral field ϕ. We just want to see that that what is the form of the kinetic terms in the low energy limit. We do this because the structure of the kinetic terms can reveal the metric on the moduli space (like the metric in the sigma model). So, we do do know that the kinetic terms are actually the terms which have derivatives of fields (except for terms that have derivative interactions). We see that in the expansion of $\bar{\phi}$ given in (2.62), the terms that can contribute to the kinetic terms are as follows;

$$-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\bar{z},\ \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi} \tag{4.18}$$

Now,in the term that we are analysing (i.e. the second term in (4.17)), only the $\theta\theta\bar{\theta}\bar{\theta}$ terms will survive and thus, from the expansion of $\mathcal{F}'(\phi)$, we only need the term that is independent of θ and $\bar{\theta}$ and the term which is linear in θ. Now, from the expansion of $\mathcal{F}'(\phi)$, the required terms are as follows;

$$\mathcal{F}'\left(z(x)+i\theta\sigma^\nu\bar{\theta}\partial_\nu z-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 z+\sqrt{2}\theta\psi-\frac{i}{\sqrt{2}}\theta\theta\partial_\nu\psi\sigma^\nu\bar{\theta}-\theta\theta f\right)=\mathcal{F}'(z)+\sqrt{2}\mathcal{F}''(z)\theta\psi+\text{other} \tag{4.19}$$

where "other" refers to other terms. Now, using (4.18) and (4.19), we can calculate the kinetic terms that come out of (4.17) as follows;

$$\text{Kinetic terms from } \bar{\phi}F'(\phi)=-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\bar{z}\mathcal{F}'(z)+\frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}\left(\sqrt{2}\psi\right)=\theta\theta\bar{\theta}\bar{\theta}\mathcal{F}''(z)\left[\frac{1}{4}|\partial_\mu z|^2-i\psi\sigma^\mu\partial_\mu\bar{\psi}\right] \tag{4.20}$$

Using (4.20) in (4.17) suggests that Im $\mathcal{F}''(\langle z\rangle)=$ Im $\mathcal{F}''(a)$ behaves like a metric on the moduli space. We can reconfirm this suggestion that we are getting by computing the first term in (4.17) as well. To do that calculation, we must look at the expression of W_α in(2.85) and see that the terms that contribute to the kinetic terms in $W^\alpha W_\alpha$ are $i(\sigma^{\mu\nu}\theta)_\alpha F_\mu$ multiplied by itself and $\theta\theta(\sigma^\mu_{\alpha\dot{\beta}}D_\mu\bar{\lambda}^{\dot{\beta}})$ multiplied to the $-i\lambda_\alpha$ term. Now, the only terms that will be non zero in the $d^2\theta$ integration is the f term (i.e. the $\theta\theta$ term). This means that in the expansion of $\mathcal{F}''(\phi)$ in it's components, only $\mathcal{F}''(z)$ will contribute non zero terms in the first term of (4.17) (and obviously, $\mathcal{F}''(z)$ will get multiplied by the f term of $W^\alpha W_\alpha$ given in (2.89)). So, the first term of (4.17) becomes;

$$\begin{aligned}&\frac{1}{16\pi}\text{Im}\int\theta\theta\, d^2\theta\mathcal{F}''(z)\left[\frac{1}{2}F^{\mu\nu}\left(i*F_{\mu\nu}-F_{\mu\nu}\right)-2i\lambda\sigma^\mu D_\mu\bar{\lambda}\right]\\&=\frac{1}{8\pi}\text{Im}\,\mathcal{F}''(z)\left[-\frac{1}{4}F^{\mu\nu}\left(F_{\mu\nu}-i*F_{\mu\nu}\right)-i\lambda\sigma^\mu D_\mu\bar{\lambda}\right]\end{aligned} \tag{4.21}$$

where I have made the constant in front of $F^{\mu\nu}F_{\mu\nu}$ term $-1/4$ to make it look like the Maxwell's lagrangian. So again, we get the strong indication that Im $\mathcal{F}''(a)$ behaves as a metric on the moduli space of vaccua. Let's denote $\mathcal{F}''(a)$ as $\tau(a)$ (we can see it as a vacuum dependant coupling). Now, Seiberg and Witten [3] and some of the reviewers for their work (for example [20]) take the metric on the moduli space as;

$$(ds)^2=\text{Im }\mathcal{F}''(a)da\,d\bar{a}=\text{Im }\tau(a)da\,d\bar{a} \tag{4.22}$$

Now, a skeptical reader might think that as a and $-\bar{a}$ are gauge equivalent, then τa must be an even function of a and the good news is that it is. For discussing this issue further, I will have to use a result from the perturbative analysis of the beta function of supersymmetric gauge theories. This result is derived in a famous 1988 paper by N. Seiberg in [19]. I won't present this derivation here as it is beyond the scope of this thesis. The derivation of this result requires the familiarity with non renormalization theorems of susy gauge theory. An interested reader may consult [2] for a quick introduction to these theorems or read section 27.6 of [21] for a more detailed treatment. In [19], the one loop perturbative result for $\mathcal{F}(\Psi)$ is calculated and non renormalization theorems state that this higher loop corrections are zero. Here, Ψ is N=2 chiral superfield (as opposed to ϕ, which is N=1 chiral field). We didn't need N=2 chiral fields until now and we won't need them in future as well. I just introduced it to quote the result that Seiberg calculated. The only thing that we need to know about it is it's first two terms. Introduce superspace coordinates $(\tilde{\theta}^\alpha, \tilde{\bar{\theta}}^{\dot{\alpha}})$ and then, the first two terms are;

$$\Psi(\tilde{y}, \tilde{\theta}) = \phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \text{last term}, \;\; \text{where } \tilde{y}^\mu = y^\mu + i\tilde{\bar{\theta}}\sigma^\mu\tilde{\theta} \tag{4.23}$$

The last term doesn't concern our work. An interested reader might consult [20] for the full expression. The one loop $\mathcal{F}(\Psi)$ is calculated to be;

$$\mathcal{F}(\Psi) = \frac{i}{2\pi}\Psi^2 \ln\frac{\Psi^2}{\Omega^2} \tag{4.24}$$

where Ω^2 is a constant related to the energy scale. It's details are not important (for an interested reader, this result is quoted in equation 22 of [19] alongwith the instanton contributions that I have not written down). Now, large a means high energy and at high energy, the leading term in Ψ^2 has to be a^2 as can be seen from the expression (4.23) and reminding ourselves that $\langle z\rangle^a \sim a^2$. Moreover, since N=2 SYM is asymptotically free, we also conclude that large a limit means weak coupling limit. In addition, since in the large a limit, $\Psi^2 \sim a^2$, we have;

$$\mathcal{F}(a) \sim \frac{i}{2\pi}a^2 \ln\frac{a^2}{\Omega^2} \;\; (a \to \infty) \tag{4.25}$$

Moroever, for $\tau(a)$, we have;

$$\tau(a) = \frac{\partial^2\mathcal{F}(a)}{\partial a^2} = \frac{\partial}{\partial a}\left[\frac{i}{\pi}\left(a\ln\frac{a^2}{\Omega^2} + 2a\right)\right] = \frac{i}{\pi}\left(\ln\frac{a^2}{\Omega^2} + 3\right) \;\; (a \to \infty) \tag{4.26}$$

Now, we can see the resolution of the potential problem that I mentioned under (4.22). We see that $\tau(a)$ is indeed an even function of a. Still, there might be a problem as complex logarithms are multivalued but actually, there is no problem because the multivaluedness of the complex logarithms lies in their imaginary part. However, since the metric in (4.22) is just lm τ, we see that only the real part of the logarithm in (4.26) contributes and thus, there is no problem of multivaluedness of the metric.

There is one important point that I need to point out before closing this section. We first prove a theorem about holomorphic functions;

Theorem 4.2.1 Let $f(z)$ be a function of the complex variable $z = x + iy$ where x, y are real. Then, if $f(z)$ is holomorphic, then it means that it's imaginary part is harmonic.

Proof: Let's write $f(z)$ as;

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{4.27}$$

Now, if $f(x,y)$ is holomorphic, then it obeys Cauchy Riemann conditions i.e;

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4.28}$$

Using the expression for z, (4.28) implies;

$$\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}} = i\left(\frac{\partial v}{\partial z} - \frac{\partial v}{\partial \bar{z}}\right), \frac{\partial u}{\partial z} - \frac{\partial u}{\partial \bar{z}} = i\left(\frac{\partial v}{\partial z} + \frac{\partial v}{\partial \bar{z}}\right) \Rightarrow \frac{\partial u}{\partial z} = i\frac{\partial v}{\partial z}, \frac{\partial u}{\partial \bar{z}} = -i\frac{\partial v}{\partial \bar{z}} \tag{4.29}$$

The last two equations are just complex conjugates of each other. The last equation in (4.29) implies that;

$$\frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = 0 \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0 \tag{4.30}$$

which is nothing but a restatement of the holomorphicity of the function $f(z)$. We will need this form in our work. Now, we calculate 'laplacian' of v (while making use of the fact that derivatives of u and v w.r.t. $\bar{z}$ vanish);

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial z^2}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 0 \tag{4.31}$$

while I used the expression for z in the last step. Now, since the laplacian of v vanishes, it means that v (i.e. $Im\, f$) is harmonic (The same reasoning can be made for u). (QED)

Now, the reader must be careful in reading the theorem as I used z for the complex coordinate but in our work, the complex coordinate is a and z is the scalar component of N=2 multiplet. Now, since $\mathcal{F}(a)$ and hence $\tau(a)$ are holomorphic, it means that their imaginary part is harmonic. Now, there is another theorem that I will state without proof. An interested reader may consult any good text on complex analysis or potential theory for the proof (see for example, section 18.6 of [22]).

Theorem 4.2.2 Let $f(x,y)$ be a non constant harmonic function on a bounded subset R of the $x-y$ plane. Then there is no local maxima or minima of f at any interior point of R. Moreover, the maxima and minima reside over the surface of R.

This theorem poses a big danger for $\tau(a)$ as a metric on the moduli space. Since there is no minima of $\tau(a)$ on **any** bounded subset of the complex plane, it can't be positive definite. Failure to be positive definite is a disaster for the metric. The only way out of this dilemma is that we choose different regions of the moduli space and have different a functions in those regions. (a have to functions of the coordinate on the complex plane i.e. u and thus I am talking about $a(u)$ functions). This is the problem that Seiberg Witten duality attempts to solve. In our effective lagrangian (4.17) we have only the ϕ, W fields. Seiberg Witten duality provides a dual set of fields (dubbed ϕ_D and W_D where D stands for dual) which is appropriate for a different region of the moduli space. This duality transformation is what we will discuss now.

4.3 Seiberg Witten duality transformation

We look at (4.22) now and we realize that this line element can be written as;

$$(ds)^2 = \frac{1}{2i}\left(d(\mathcal{F}'(a))d\bar{a} - d(\bar{\mathcal{F}}'(\bar{a}))da\right) = \frac{1}{2i}\left(d(a_D)d\bar{a} - d(\bar{a}_D)da\right) = \text{Im}\,(d\bar{a}\,da_D) \text{ where } a_D = \mathcal{F}'(a) \tag{4.32}$$

Now, it is easily seen that (4.32) is symmetric in a and a_D (i.e. it will have the same general form if we swap $a \leftrightarrow a_D$). This realisation is at the heart of Seiberg Witten duality. It means that we can use a_D as the fundamental parameter and then, the metric will have the same form as (4.22) with a different τ (i.e. $\tau_D(a_D)$). The subscript D corresponds to "dual". Now, this duality can be extended to the whole chiral field ϕ as;

$$\phi_D = \mathcal{F}'(\phi) \tag{4.33}$$

We have two fields i.e. ϕ and ϕ_D and we also see that derivative of the prepotential $\mathcal{F}(\phi)$ is ϕ_D. Now, we can look for a function (let's call this function $-\mathcal{F}_D(\phi_D)$ and the reason for minus sign will be clear later) whose derivative gives us ϕ. This function can easily be found by a legendre transformation as follows;

$$\mathcal{F}_D(\phi_D) = \mathcal{F}(\phi) - \phi\phi_D \tag{4.34}$$

It is easy to see that $\mathcal{F}'_D(\phi_D) = -\phi$ and this is the dual prepotential corresponding to the duality transformation (4.33).Let's call this duality transformation S for future reference i.e.;

$$S : \phi \to \phi_D = \mathcal{F}'(\phi),\, \mathcal{F}(\phi) \to \mathcal{F}_D(\phi_D) \tag{4.35}$$

Now, we can see that how does the second term of (4.17) transforms under S. The transformation is;

$$\text{Im}\int d^2\theta d^2\bar{\theta}\,\bar{\phi}\mathcal{F}'(\phi) \to \text{Im}\int d^2\theta d^2\bar{\theta}\,\bar{\phi_D}\mathcal{F}'_D(\phi_D) = \text{Im}\int d^2\theta d^2\bar{\theta}\,\bar{\mathcal{F}}'(\bar{\phi})(-\phi) = \text{Im}\int d^2\theta d^2\bar{\theta}\,\bar{\phi}\mathcal{F}'(\phi) \tag{4.36}$$

where the last step, I used the fact that $\text{Im}\, z = -\text{Im}\,\bar{z}$. We see that the second term of (4.17) is invariant under the duality transformation.

We can now ask the question that is this a coincidence or is this something that we could have known beforehand. The answer to this question is that we could have known this before hand and to see this, we define a multiplet $a^\beta = (a_D, a)$ and using this definition, the metric in (4.32) is written as;

$$(ds)^2 = -\frac{i}{2}(da_D d\bar{a} - d\bar{a}_D da) = -\frac{i}{2}(\epsilon^{12} da^1 d\bar{a}^2 + \epsilon^{21} da^2 d\bar{a}^1) = -\frac{i}{2}\epsilon^{\alpha\beta} da^\alpha d\bar{a}^\beta \tag{4.37}$$

It means that the metric will be invariant under any transformation of $a^\alpha \to M^\alpha_\beta a^\beta$ (where M is a matrix) such that $\epsilon^{\alpha\beta}$ is an invariant if it was transformed by M. However, due to the presence of $d\bar{a}^\beta$, it is necessary that the matrix M should not be effected by complex conjugation. These are the well known properties of $SL(2,\mathbb{C})$ matrices (remember that the matrices under which the Weyl spinors ϕ_α transform are $SL(2,\mathbb{C})$ matrices. Invariance of $\epsilon^{\alpha\beta}$ under the $SL(2,\mathbb{C})$ matrices is the reason because of which we raise and lower the spinor index by $\epsilon^{\alpha\beta}$). This transformation can be extended to ϕ^α.

We will see that in order for the matrix $U(b)$ to correspond to a symmetry, b should be an integer and thus, the group that we should consider is $SL(2,\mathbb{Z})$. Now, if we look at the generators of $SL(2,\mathbb{Z})$, they are

$$U(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \forall\, b \in \mathbb{Z}\ , S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.38}$$

Let's take take the S transformation (we will later talk about the $U(b)$ case). The S case gives us the following transformation on ϕ^α;

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} = \begin{pmatrix} \phi \\ -\phi_D \end{pmatrix} \tag{4.39}$$

Under (4.39) transformation, we see that the second term of (4.17) transforms as;

$$\text{Im} \int d^2\theta d^2\bar{\theta}\, \bar{\phi}\mathcal{F}'(\phi) = \text{Im} \int d^2\theta d^2\bar{\theta}\, \bar{\phi}\phi_D \to -\text{Im} \int d^2\theta d^2\bar{\theta}\, \bar{\phi}_D\phi = \text{Im} \int d^2\theta d^2\bar{\theta}\, \bar{\phi}\phi_D \tag{4.40}$$

So, this invariance of second term in (4.17) under the duality transformation is something that we could have inferred beforehand by requiring the invariance of the metric (4.32).

Now, we need to determine the appropriate duality transformation for the term in (4.17). For this aim, we need to remind ourselves that $SU(2)$ gauge symmetry has broken to an abelian gauge and thus, the expression for W_α given in (2.78) is reduced to the first term only. We thus have the electromagnetic field tensor $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ which satisfies the bianchi identity $\epsilon^{\alpha\beta\rho\sigma}\partial_\beta * F_{\rho\sigma} = 0$. We can ask that what is the corresponding equation in the superspace? i.e. what is the constraint on W_α? This constraint turns out to be $\text{Im}(D_\alpha W^\alpha) = 0$ as can be checked by using (2.85) (but replacing D_μ by ∂_μ as we are dealing with abelian theory here) and (2.79). Now, the trick to figure out the duality transformation on the first term of (4.17) is to invoke the partition function calculated for the V field subject to the constraint that $\text{Im}(D_\alpha W^\alpha) = 0$. We do this by using the common technique of lagrange multipliers. We introduce another field $V_D/2$ as the lagrange multiplier (the reason for this notation will become clear later). Now, the partition function is just

$$\text{partition function} = \int \mathcal{D}V \, \exp\left[i\mathcal{L}\right]$$

$$= \int \mathcal{D}W\mathcal{D}V_D \, \exp\left[\frac{i}{16\pi}\text{Im}\int d^4x \left(\int d^2\theta \mathcal{F}''(\phi)W^\alpha W_\alpha + \frac{1}{2}\int d^2\theta d^2\bar{\theta} V_D D_\alpha W^\alpha\right)\right] \tag{4.41}$$

Now, we consider the second term and manipulate it as follows;

$$\int d^2\theta d^2\bar{\theta} V_D D_\alpha W^\alpha = -\int d^2\theta d^2\bar{\theta} D_\alpha V_D W^\alpha = -\int d^2\theta \left(\bar{D}^2 D_\alpha V_D\right) W^\alpha = -4\int d^2\theta (W_D)_\alpha W^\alpha \tag{4.42}$$

The steps in (4.42) needs explaining. First step uses the fact that;

$$\int d^2\theta d^2\bar{\theta} D_\alpha(F) = 0 \tag{4.43}$$

where F is any superfield. The result (4.43) can be understood by the fact that in the $\theta\theta\bar{\theta}\bar{\theta}$ integral, only $\theta\theta\bar{\theta}\bar{\theta}$ terms survive but $D_\alpha F$ doesn't have any $\theta\theta\bar{\theta}\bar{\theta}$ term. In the second step, we use the fact that operating $d^2\bar{\theta}$ integral is the same as operating $\bar{D}^2$ on any superfield. Both operators give non zero result for $\theta\theta\bar{\theta}\bar{\theta}$ terms or $\bar{\theta}^2$ terms only. Moreover, since W_α is chiral, we can take it out of the operation of $\bar{D}^2$ operator. The last step defines W_D i.e.;

$$W_D = -\frac{1}{4}\bar{D}^2 D_\alpha V_D \tag{4.44}$$

Now, we see the reason because of which we called the lagrange multiplier V_D. Now, the partition function in (4.41) becomes;

$$\int \mathcal{D}V_D \int \mathcal{D}W \, \exp\left[\int d^4x d^2\theta \left(\frac{i}{16\pi} \, \text{Im} \, \mathcal{F}''(\phi)W^\alpha W_\alpha + \frac{i}{8\pi} \, \text{Im} \, W_D^\alpha W_\alpha\right)\right] \tag{4.45}$$

Now, we do the gaussian functional integral of W using the identity;

$$\int_\infty^\infty dx e^{-\alpha^2 x^2 + \beta x} = \frac{1}{\alpha}\sqrt{\pi} e^{\beta^2/4\alpha^2} \tag{4.46}$$

the well known result after the gaussian integration is;

$$\int \mathcal{D}V_D \, \exp \left[\frac{i}{16\pi} \, \mathrm{Im} \int d^4x \, d^2\theta \left(-\frac{1}{\mathcal{F}''(\phi)} W_D^\alpha (W_D)_\alpha \right) \right] \tag{4.47}$$

where I ignored an irrelevant factor in front of the exponential (i.e. the factor doesn't depend on W). This shows that for the dual field W_D^α, we have a dual coupling τ_D i.e.;

$$\tau_D(\phi_D) = -\frac{1}{\tau(\phi)} = -\frac{1}{\mathcal{F}''(\phi)} = -\frac{d\phi}{d\mathcal{F}'(\phi)} = -\frac{d\phi}{d\phi_D} = \frac{d}{d\phi_D} \left(\mathcal{F}_D'(\phi_D) \right) = \mathcal{F}_D''(\phi_D) \tag{4.48}$$

So, we get $\tau_D(\phi_D) = -\tau^{-1}(\phi) = \mathcal{F}_D''(\phi_D)$. It shows that if $\tau(\phi)$ is small then the dual transformation takes us to strong coupling. We have seen this transformation before in chapter 3 in Olive Montonen duality. This shows the connection between Olive Montonen duality and Seiberg Witten duality. At the end, we get the result of the duality transformation on the lagrangian in (4.17) as;

$$\mathcal{L}_{N=2}(\text{effective}) = \frac{1}{16\pi} \, \mathrm{Im} \, \left[\int d^2\theta \mathcal{F_D}''(\phi_D) W_D^\alpha (W_D)_\alpha + \int d^2\theta d^2\bar{\theta} \bar{\phi}_D \mathcal{F}_D'(\phi_D) \right] \tag{4.49}$$

Now, we can talk about the $U(b)$ generator which I mentioned under (4.38). Before talking about that transformation, I recall that for an arbitrary complex number z, we have;

$$\mathrm{Im}\, z = \frac{z - \bar{z}}{2i} \tag{4.50}$$

Using (4.50), and using the fact that $\phi_D = \mathcal{F}'(\phi)$, we can write (4.49) as;

$$\mathcal{L}_{N=2}(\text{effective}) = \frac{1}{16\pi} \, \mathrm{Im} \, \int d^2\theta \frac{d\phi_D}{d\phi} W_D^\alpha (W_D)_\alpha + \frac{1}{32\pi i} \int d^2\theta d^2\bar{\theta} (\bar{\phi}\phi_D - \bar{\phi}_D \phi) \tag{4.51}$$

Now, let's consider the transformation induced by $U(b)$ given in (4.38) (remember that b is real). It will be clear that b should be an integer. The corresponding transformation is;

$$\phi_D \to \phi_D + b\phi, \, \phi \to \phi \tag{4.52}$$

This gives;

$$\bar{\phi}\phi_D - \bar{\phi}_D \phi \to \bar{\phi}(\phi_D + b\phi) - (\bar{\phi}_D + b\bar{\phi})\phi = \bar{\phi}\phi_D - \bar{\phi}_D \phi \tag{4.53}$$

Thus, the second term in (4.51) is invariant. Moreover, we have;

$$\frac{d\phi_D}{d\phi} \to \frac{d\phi_D}{d\phi} + b\frac{d\phi}{d\phi} = \frac{d\phi_D}{d\phi} + b \tag{4.54}$$

So, the lagrangian in (4.51) transforms as;

$$\mathcal{L}_{N=2}(\text{effective}) \to \mathcal{L}_{N=2}(\text{effective}) + \frac{b}{16\pi} \mathrm{Im} \int d^2\theta W_D^\alpha (W_D)_\alpha \tag{4.55}$$

Now, we need to remind ourselves that in the calculation of the functional integrals, the action's exponential appears (i.e. e^{iS}) and thus, the actions differing by $2\pi n, n \in \mathbb{Z}$ should be physically equivalent. In other words, any transformation that maps the action $S \to S + 2\pi n$, $n \in \mathbb{Z}$ can be considered as a symmetry. We can see that $U(b)$ for integer b is such a transformation. We can see that (4.55) will add the following term to the action;

$$\frac{b}{16\pi} \mathrm{Im} \int d^4x \int d^2\theta W_D^\alpha (W_D)_\alpha = -\frac{b}{16\pi} \int d^4x F^{\mu\nu} * F_{\mu\nu} = -2\pi b n \tag{4.56}$$

where

$$n = \frac{1}{32\pi^2}\int d^4xF^{\mu\nu} * F_{\mu\nu} \text{ is the Yang Mills instanton number} \tag{4.57}$$

We know that n is an integer and contributes to the CP violating θ term that we had in (3.29). For a reader who doesn't know why n is an integer it is sufficient to know that this term is a topological invariant and it can be normalised to be an integer. Further discussion about this term requires the concept of winding numbers and homotopy classes and I will not present it here. An interested reader might consult [41] for a brief discussion of instantons and the more detailed reference for instantons would be [42].

So, the term that adds up to the action (given in (4.56)) is indeed an integer and thus, $U(b)$, $b \in \mathbb{Z}$ is a symmetry transformation. Since S and $U(b)$ in (4.38) are both symmetry transformations of the lagrangian in (4.17), we can conclude that Seiberg Witten duality group is $SL(2,\mathbb{Z})$. We can contrast this with the Olive Montonen duality case. The $U(b)$ transformation here corresponds to the $\theta \to \theta + 2\pi$ transformation in the Olive Montonen case and the S transformation here corresponds to the $\tau \to -\tau^{-1}$ transformation in the Olive Montonen case. Both duality groups are $SL(2,\mathbb{Z})$. So, we see the relationship between two duality transformations now.

4.4 Moduli space's singularities and thier interpretation

Here, singularities refer to the branch points of the function $a(u)$ and/or $a_D(u)$. In other words, singularity here refers to the point on the moduli space (which is a complex u plane for our problem) such $a(u)$ or (and) $a_D(u)$ doesn't (don't) return to their original values when taken around an arbitrary loop around such a point. We say that such a loop has a **non trivial monodromy** for $a(u)$ and/or $a_D(u)$. Identifying such points on the moduli space will be fruitful as we shall see.

4.4.1 One obvious singularity

One singularity can be easily identified. We do have the expression for $\mathcal{F}(a)$ as $a \to \infty \Rightarrow u \to \infty$ (as $u = a^2/2$) in (4.25). Now, since $a_D = \mathcal{F}'(a)$, we use (4.25) to get;

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a} \sim \frac{\partial}{\partial a}\left[\frac{i}{2\pi}a^2 \ln\frac{a^2}{\Omega^2}\right] = \frac{i}{\pi}a\left(\ln\frac{a^2}{\Omega^2} + 1\right) \tag{4.58}$$

Now, if we take a circle with very large radius then going around that circle counterclockwise will correspond to $u \to ue^{2\pi i}$ and since $u = a^2/2$, it means that $a \to e^{i\pi}a = -a$. So, using (4.58), we have;

$$a_D \to \frac{i}{\pi}(-a)\left(\ln\frac{e^{2\pi i}a^2}{\Omega^2} + 1\right) = -\frac{i}{\pi}a\left(\ln\frac{a^2}{\Omega^2} + 1\right) + 2a = -a_D + 2a \tag{4.59}$$

So, we can summarise this transformation with a matrix M_∞ (called the monodromy matrix) acting on the doublet $[a_D \quad a]^T$ as follows;

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -a_D + 2a \\ -a \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{4.60}$$

4.4.2 Other singularities

I need to use the existence of $U(1)_R$ symmetry in order to proceed. R symmetry is just a transformation that mixes the susy generators into each other (i.e. Q^I_α's) and this symmetry preserves the susy lagrangian. An interested reader may consult any good resource on susy for a brief introduction to R symmetry (for example, [43])

Now, under $U(1)_R$ symmetry, the required transformations are as follows;

$$z \to e^{2i\xi} z,\ W_\alpha \to e^{i\xi} W_\alpha,\ \theta(\bar{\theta}) \to e^{i\xi}\theta(\bar{\theta}),\ d^2\theta(d^2\bar{\theta}) \to e^{-2i\xi} d^2\theta(d^2\bar{\theta}),\ \xi \in \mathbb{R} \tag{4.61}$$

We need to recall the relationship between grassman integration and grassman differentiation in order to understand the final result. Now, using (4.23) (for example, using the $\tilde{\theta}^\alpha W_\alpha$ term), we can see that $\Psi \to e^{2i\xi}\Psi$ and using (4.24), we see that $\mathcal{F}(\Psi) \to e^{4i\xi}\mathcal{F}(\Psi)$ (as $\mathcal{F}(\Psi) \propto \Psi^2$). Now, under (4.24), I wrote that I am not writing the instanton corrections to the prepotential and referred the reader to the equation 22 of [19]. However, we need the form of the instanton corrections now in order to prove that the singularity at infinity is not the only infinity and I will quote the relevant form of the instanton corrections to the prepotential $\mathcal{F}(\Psi)$ from [19]. The form of such corrections is;

$$\mathcal{F}(\Psi)_{\text{Instantons}} = \Psi^2 \sum_{j=1}^{\infty} \mu_j \left(\frac{\Omega^2}{\Psi^2}\right)^{2j},\ \text{ where } \mu_j \text{ are coefficients.} \tag{4.62}$$

Now, we have;

$$\mathcal{F}(\Psi) \to e^{4i\xi}\Psi^2 \sum_{j=1}^{\infty} \mu_j \left(\frac{\Omega^2}{\Psi^2}\right)^{2j} e^{-8j\xi} \tag{4.63}$$

Now, the symmetry $\mathcal{F} \to e^{4i\xi}\mathcal{F}$ is still intact if 8ξ is an integer multiple of 2π and thus, we have;

$$\xi = \frac{2\pi j}{8},\ j \in \mathbb{Z} \tag{4.64}$$

This refers to $\mathbb{Z}_8$ symmetry. So, the $U(1)_R$ symmetry breaks down to $\mathbb{Z}_8$ symmetry. Under this symmetry, we have;

$$z^2 \to e^{4i\xi j} z^2 = e^{i\pi j} z^2 = -z^2 \text{ if j is odd} \tag{4.65}$$

Since $z^2 \to -z^2$ has to be a symmetry, it means that $u \to -u$ has to be a symmetry and thus, $u = \tilde{u}$ is a singularity then $u = -\tilde{u}$ has to be a singularity. On the other hand, the singularity can be a fixed point of the symmetry $u \to -u$. Such fixed points are 0 and $|u| \to \infty$. Now, the question arises that is $u = 0$ a singularity? Well, considering a contour on Riemann sphere going around $u = 0$ can be seen to have the same monodromy action as the contour around $|u| \to \infty$ point (can be seen by contour deformation and by using the fact that we are assuming no other singularities besides at $u = 0$ and $|u| \to \infty$). Thus, the monodromy matrices for these two singularities are the same. Since $a \to -a \Rightarrow a^2 \to a^2$ under M_∞, we can see that the same conclusion holds for the monodromy around zero and thus, a^2 is a perfectly good coordinate. If this is the case, then the equation $u = a^2/2$ (which was initially true for large $|u|$ only) can be taken to hold on the whole u plane now and if this is case, then the expression for $\tau(a)$ that was initially true for large $|u|$ only can be taken to hold on the entire u plane now. However, as noted earlier, $\text{Im}\tau(a)$ is a holomorphic function and thus, it can't be positive definite on the entire u plane. So, we can't have only two singularities.

We can now consider three singularities. Since the argument that I gave above (which involved the breaking of $U(1)_R$ symmetry to $\mathbb{Z}_8$ symmetry on the whole u plane), the non zero singularities (apart from $|u| \to \infty$ singularity) should come in pairs and thus, we can have three singularities as $|u| \to \infty, \tilde{u}, -\tilde{u}$.

4.4.3 An aside: Massive SUSY representations

We talked about massless representations in section 2.2 but we need some small facts and the concept of short representations (i.e. BPS representations with reference to the central charges) in order to proceed. We can have this discussion for any number of supersymmetries but I will just restrict myself to $N = 2$ susy. Now, if we have a massive representation (with mass m) then we can go to the rest frame of the particle and the momentum generator is;

$$P_\mu = (m, 0, 0, 0) \tag{4.66}$$

Now, the anti commutation relations are;

$$\{Q^I_\alpha, (Q^J_\beta)^\dagger\} = 2m\delta_{\alpha\beta}\delta^{IJ},\ \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ},\ \{(Q^I_\alpha)^\dagger, (Q^J_\beta)^\dagger\} = \epsilon_{\alpha\beta}Z^{*IJ} \tag{4.67}$$

Now, we use the fact that Z^{IJ} is an anti symmetric matrix and thus, it can be written as;

$$Z^{IJ} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \tag{4.68}$$

where Z is positive. Now, I define the following operators;

$$A_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger),\ B_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger) \tag{4.69}$$

and using (4.67), we have;

$$\{A_\alpha, (A_\beta)^\dagger\} = \frac{1}{2}\{Q^1_\alpha + \epsilon_{\alpha\rho}(Q^2_\rho)^\dagger, (Q^1_\beta)^\dagger + (\epsilon_{\beta\sigma}(Q^2_\sigma)^\dagger)^\dagger\} = (2m - Z)\delta_{\alpha\beta} \tag{4.70}$$

We can calculate $\{B_\alpha, (B_\beta)^\dagger\}$ as well but we don't need that for our argument (the reader can check that it will give the same result with a plus instead of minus sign). (4.70) shows that (due to positivity of Hilbert space) A_α should be trivial on the Hilbert space if $2m - Z \leq 0$. If this is the case, then the length of a massive multiplet is the same as the massless multiplets and such massive multiplets are known as **short** multiplets. If $2m - Z > 0$ then the corresponding multiplets are known as long multiplets. If this bound is saturated (i.e. $2m = Z$) then this multiplet is known as BPS multiplet and the corresponding states are known as BPS states. The mass of BPS states is thus predictable using the central charge information.

4.4.4 Other singularities: Continued

We want to interpret the other singularities (i.e. we want to find their origin). We can take a hint from our failed attempt to declare $u = 0$ as a distinct singularity. At $u = 0$, the gauge group is restored to $SU(2)$ and thus, all the gauge bosons are massless. We can take this hint and ask that is it the case that at the singularities $u = \pm\tilde{u}$, some additional gauge bosons become massless? Well, the answer to this question is no and the justification for this answer requires understanding of (and a single result from) superconformal invariance which is beyond the scope of this work. Loosely stated, for massless gauge bosons, the dimension of the operator $Tr\phi^2$ should be zero but it is 2 and thus, we have a contradiction. An interested reader may consult [3] for some extra detail. With the option of massless gauge bosons out of the way, another option to consider (which is by no means very obvious) is to consider massless collective excitations (i.e. massless monoples).

We should keep in mind that the fields that become massive by the higgs mechanism should be in the short multiplets as the higgs mechanism can't produce enough helicity states to send the massive states in a long multiplet (the long multiplet has a length of $2^{2(2)} = 16$ while the short multiplet has a length of $2^2 = 4$ and thus, the remaining 12 helicity states can't be generated by Higg's mechanism). Thus, the massive representation should be a BPS state and thus, their mass is predictable. Now, before proceeding, I need to rewrite here (as I did write under (2.5)) that for our purpose, the classical central charges are calculated by E. Witten and D. Olive in [4]. I won't go in the details here (as they are beyond the scope of this work) but the quantum mechanical generalisation can be made (following Seiberg and Witten [3]) by coupling the $N = 2$ SYM theory with $N = 2$ hypermultiplet (which is made by two $N = 1$ chiral multiplets say, M and $\tilde{M}$ and they should be BPS multiplets). The hypermultiplets correspond to electrically charges particles and the central charge is $Z = an_e$ where n_e is the electrical charge. The duality transformation gives $Z = a_D n_m$ (where n_m is magnetic charge) for the monopoles. Now, the point on the moduli space where the monopoles are massless corresponds to the point where $a_D = 0$. Moreover, the total central charge is $Z = an_e + a_D n_m$. Now, coupling of the dual chiral field in $N = 2$ SYM with the dual hypermultiplets correspond to SQED theory and it's beta function is given as (calculated in [23]);

$$\Lambda \frac{d\lambda_D}{d\Lambda} = \frac{\lambda_D^3}{8\pi^2} \tag{4.71}$$

where λ_D is the coupling and Λ is the energy scale (the subscript D refers to the fact that we are talking about the dual theory i.e. monopoles are coupled to ϕ_D and W_D). Now, since the energy scale is proportional to $\langle z_D \rangle$ which is proportional to a_D, we can use a_D in place of Λ. Now, using the definition of τ (and hence of τ_D) from section 3.7 (for zero θ angle and for the coupling now denoted by λ_D), we have;

$$a_D \frac{d\tau_D}{da_D} = a_D \frac{d}{da_D}\left(\frac{4\pi i}{\lambda_D^2}\right) = -\frac{8\pi i}{\lambda_D^3} a_D \frac{d\lambda_D}{da_D} = -\frac{i}{\pi} \Rightarrow \tau_D = -\frac{i}{\pi}\ln\, a_D \tag{4.72}$$

where I used (4.71) and set the integration constant to zero. Using (4.48), we see that;

$$\frac{da}{da_D} = \frac{i}{\pi}\ln\, a_D \Rightarrow a = \tilde{a} + \frac{i}{\pi}(a_D \ln\, a_D - a_D) \sim \tilde{a} + \frac{i}{\pi} a_D \ln\, a_D \tag{4.73}$$

where I dropped the last bracketed term because it is small as compared to the first bracketed term for small a_D and $\tilde{a}$ is a constant. Remember that we are solving for the region around $u = \tilde{u}$ where $a_D = 0$. Now, we need to know a_D and a in the vicinity of $u = \tilde{u}$ in order to derive the monodromy matrix. We use the Taylor's series on a_D (and use the fact that $a_D(u = \tilde{u}) = 0$) and use (4.73) to get;

$$a_D(u) = a_D(\tilde{u}) + \frac{\partial a_D}{\partial u}|_{u=u_0}(u - \tilde{u}) + \mathcal{O}(|u - \tilde{u}|^2) = \zeta(u - \tilde{u}), \text{ where } \zeta = \frac{\partial a_D}{\partial u}|_{u=u_0} \tag{4.74}$$

$$a(u) = \tilde{a} + \frac{i}{\pi}\zeta(u-\tilde{u})\ln\,(u-\tilde{u}) \tag{4.75}$$

Now, going around the point $u = \tilde{u}$ corresponds to $u - \tilde{u} \to e^{2\pi i}(u-\tilde{u})$ and thus, we have;

$$a_D(u) \to \zeta e^{2\pi i}(u-\tilde{u}) = e^{2\pi i}a_D = a_D \tag{4.76}$$

$$a \to \tilde{a} + \frac{i}{\pi}\zeta e^{2\pi i}(u-\tilde{u})\ln\,[e^{2\pi i}(u-\tilde{u})] = a - 2\zeta(u-\tilde{u}) = a - 2a_D \tag{4.77}$$

So, we have the following monodromy matrix (i.e. $M_{\tilde{u}}$)

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a - 2a_D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = M_{\tilde{u}}\begin{pmatrix} a_D \\ a \end{pmatrix} \tag{4.78}$$

To find the monodromy matrix at $u = -\tilde{u}$, we employ the same observation that we used while discussing the case of two singularities. There, we observed that the monodromy matrices at infinity and at $u = 0$ should be the same (we used the contour deformation argument and the absence of a third singularity). The same thing can be done here in order to determine the monodromy matrix at $u = -\tilde{u}$. We observe that if we make a loop that goes around $u = \tilde{u}$ and $u = -\tilde{u}$ then the monodromy matrix due to this loop should be the same as M_∞ (the counter deformation argument can again be used here to establish the equivalence of this counter to a counter made around the point at infinity). There is one question here i.e. "which point should we circle first?". Well, the answer to this question is that different orderings will correspond to different dyons as we shall see below (recall that a dyon is a state with electrical and magnetic charge). For now, let's consider both orderings. Here they are (we will need to use (4.60) and (4.78) for M_∞ and $M_{\tilde{u}}$);

$$M_\infty = M_{\tilde{u}}M_{-\tilde{u}} \Rightarrow M_{-\tilde{u}} = M_{\tilde{u}}^{-1}M_\infty = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{4.79}$$

$$M_\infty = M_{-\tilde{u}}M_{\tilde{u}} \Rightarrow M_{-\tilde{u}} = M_\infty M_{\tilde{u}}^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \tag{4.80}$$

Now, recall that the singularity at $u = \tilde{u}$ corresponded to the monopoles becoming massless as $a_D = 0$ at that point. Can we come up with another method to verify this fact that also allows us to find the interpretation of the singularity at $u = -\tilde{u}$? Well, actually we can and the method is an obvious one. Look at the formula for the central charge just above (4.71). It can be written as;

$$Z = n_e a + n_m a_D = \begin{pmatrix} n_m & n_e \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} \tag{4.81}$$

Now, under a monodromy action with monodromy matrix M, we have;

$$Z \to \begin{pmatrix} n_m & n_e \end{pmatrix} M \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \tilde{n}_m & \tilde{n}_e \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} \text{ where } \begin{pmatrix} \tilde{n}_m & \tilde{n}_e \end{pmatrix} = \begin{pmatrix} n_m & n_e \end{pmatrix} M \tag{4.82}$$

Now, if a particular singularity corresponds to a particular (n_e, n_m) state becoming massless, then the monodromy action of a contour around that singularity should render $\begin{pmatrix} \tilde{n}_m & \tilde{n}_e \end{pmatrix} = \begin{pmatrix} n_m & n_e \end{pmatrix}$ i.e. $\begin{pmatrix} n_m & n_e \end{pmatrix}$ should be a left eigenvector of the monodromy matrix with unit eigenvalue. Employing this criteria to something we already know (i.e. the singularity at $M_{\tilde{u}}$ corresponds to the monopoles becoming massless), we can check the validity of this criteria as follows;

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{4.83}$$

So, we see that $(n_e, n_m) = (0, 1)$ state indeed satisfies this criteria.

4.4.5 An aside: Stable BPS states

I am taking a digression in order to address the criteria for a dyon state to be stable in order to avoid a potential confusion in the future (adopting the argument from [3]). Suppose that we have a BPS dyon state (n_e, n_m) with central charge $Z = n_e a + n_m a_D$. Since it is a BPS state, it's mass is $m = |Z|/2$. Suppose that it decays into a set of states with central charges $Z_j = (n_e)_j a + (n_m)_j a_D$ and the mass bound for massive susy states says that $m_j \leq |Z_j|/2$. Now, the conservation of charge gives us $Z = \sum_j Z_j$. Visualizing Z and Z_j's as vectors in two dimensional $n_e n_m$ space (which requires that a_D/a is not real because if it is real then the space in which (n_e, n_m) vector lives becomes one dimensional), we use the triangle inequality to get;

$$|Z| \leq \sum_j |Z_j| \tag{4.84}$$

but since $|Z| = 2m$ and $|Z_j| \leq 2m_j$, have;

$$m \leq \sum_j m_j \tag{4.85}$$

The $m < \sum_j m_j$ case will violate the conservation of energy (consider the decay in the rest frame of the decaying dyon) and thus, we can only have $m = \sum_j m_j$ at best (this means that the decay is neutrally stable). This implies that all the Z_j vectors are aligned with Z. This implies that;

$$\frac{n_e}{n_m} = \frac{(n_e)_j}{(n_m)_j} \ \forall j \tag{4.86}$$

This implies that $n_e = k(n_e)_j, n_m = k(n_m)_j$ for some integer k ($k \neq 1$ as it will imply that the decay hasn't happened). Thus, it means that n_e and n_m aren't relatively prime. So, the (n_e, n_m) states for which n_e and n_m aren't relatively prime are neutrally stable against the decay and the states (n_e, n_m) for which n_e and n_m are relatively prime are stable against the decay.

4.4.6 Other singularities: Continued

Now, we can find the left eigenvector of the monodromy matrix given in (4.79) to find the appropriate dyon state that becomes massless at $u = -\tilde{u}$. We have;

$$\begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} n_m & n_e \end{pmatrix} \Rightarrow n_e = -n_m \tag{4.87}$$

The above condition, combined with the condition that stable BPS states should have n_e and n_m relatively prime gives us $(n_e, n_m) = (-1, 1)$ as a state. So, this state becomes massless at $u = -\tilde{u}$ for the monodromy matrix (4.79).

Similarly, we can find the state that becomes massless at $u = -\tilde{u}$ for the monodromy matrix (4.80) as follows;

$$\begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} n_m & n_e \end{pmatrix} \Rightarrow n_e = n_m \tag{4.88}$$

Setting n_e and n_m relatively prime, we get $(n_e, n_m) = (1, 1)$ as the required state. So, we see that the different monodromy matrices $M_{-\tilde{u}}$ correspond to different dyon states becoming massless at $u = -\tilde{u}$.

4.4.7 Other dyons

Before closing this section, I need to point out the fact that the electrical charge in the dyon states that correspond to the points $u = \pm\tilde{u}$ are not unique. We can see that $(n_e, n_m) = (k, 1)$ with integer k still correspond to the states which have n_e and n_m relatively prime. To start, we need to find the monodromy matrix for which $\begin{pmatrix} n_m & n_e \end{pmatrix}$ is a left eigenvector with unit eigenvalue. We proceed to have;

$$\begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} n_m & n_e \end{pmatrix} \Rightarrow C = \frac{n_m(1-A)}{n_e},\, B = \frac{n_e(1-D)}{n_m} \tag{4.89}$$

In order for C and B to be integers, we have;

$$1 - A = \xi n_e \Rightarrow A = 1 - \xi n_e,\, C = \xi n_m \tag{4.90}$$

$$1 - D = \zeta n_e \Rightarrow D = 1 - \zeta n_m,\, B = \zeta n_e \tag{4.91}$$

for some functions ξ and ζ of n_e and n_m i.e. $\xi(n_e, n_m)$ and $\zeta(n_e, n_m)$. So, the matrix becomes;

$$\begin{pmatrix} 1 - \xi n_e & \zeta n_e \\ \xi n_m & 1 - \zeta n_m \end{pmatrix} \tag{4.92}$$

Now, we use the monodromy matrices we already know. For example, for $(n_e, n_m) = (0, 1)$, the monodromy matrix is;

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \Rightarrow \xi(0,1) = -2,\, \zeta(0,1) = 0 \tag{4.93}$$

Similarly, for $(n_e, n_m) = (-1, 1)$, the monodromy matrix is;

$$\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \Rightarrow \xi(-1,1) = -2,\, \zeta(-1,1) = -2 \tag{4.94}$$

This suggests that ξ is independent of n_e and suitable functions are (such that ξ and ζ are integers) $\xi = -2n_m$, $\zeta = 2n_e$. This gives us the following matrix;

$$\begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix} \tag{4.95}$$

Now, to see the presence of other states in correspondence to the singularities, we can go around the singularities in a peculiar fashion. Firstly, let me motivate two related results which can be proved in a very straightforward fashion using induction. Using (4.60), we have;

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = (-1)^1 \begin{pmatrix} 1 & -2\times(1) \\ 0 & 1 \end{pmatrix}, \; M_\infty^2 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} = (-1)^2 \begin{pmatrix} 1 & -2\times 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow M_\infty^k = (-1)^k \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \Rightarrow M_\infty^{-k} = (-1)^k \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \tag{4.96}$$

where k is a positive integer. Now, if we go around infinity k times and then, we go around $u = \tilde{u}$ and at the end, we go around infinity in the opposite sense then the corresponding monodromy matrix is;

$$M_\infty^{-k} M_{\tilde{u}} M_\infty^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-4k & 8k^2 \\ -2 & 1+4k \end{pmatrix} \tag{4.97}$$

Comparing this matrix with the matrix in (4.95), we see that matrix in (4.97) corresponds to $(n_e, n_m) = (-2k, 1)$. So, the electrical charge can be any even integer for the state that becomes massless at the point $u = \tilde{u}$. Similarly, going around the point $u = -\tilde{u}$ in the same peculiar fashion (using the monodromy matrix in (4.79)), we get;

$$M_\infty^{-k} M_{-\tilde{u}} M_\infty^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1-4k & 8k^2+2k+8 \\ -2 & 4k+3 \end{pmatrix} \tag{4.98}$$

Again, comparing this matrix with the matrix in (4.95), we see that matrix in (4.98) corresponds to $(n_e, n_m) = (-2k-1, 1)$. So, the electrical charge can be any odd integer for the state that becomes massless at the point $u = -\tilde{u}$.

4.5 Form of $a(u)$ and $a_D(u)$

We start with the following second order differential equation (z is a complex number);

$$\frac{d^2\psi(z)}{dz^2} - V(z)\psi(z) = 0 \tag{4.99}$$

Now, let $V(z)$ have poles at some finite points and also at infinity. A regular singular point $z = z_0$ for the equation (4.99) is a point such that $(z-z_0)^2 V(z)$ is analytic at $z = z_0$. Now, in order to proceed, I will quote a well known theorem without proof;

Theorem 4.5.1 *For (4.99), non trivial constant monodromy matrix arises only if the poles of $V(z)$ are regular singular points.*

Now, we know that a and a_D have three singular points i.e. $u = \pm\tilde{u}$ and $|u| \to \infty$. Following Seiberg and Witten, we set $\tilde{u} = 1$ for simplicity (in order to recover $\tilde{u}$, just replace $u \pm 1 \to (u/\tilde{u}) \pm 1$). Thus, the singular points of a, a_D become $\pm 1, \infty$. Now, the trick is to realise a and a_D as solutions of a equation like (4.99) with

$V(z)$ having poles at $\pm 1, \infty$ which are regular singular points. A general potential with these requirements is as follows;

$$V(z) = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z+1)^2} + \frac{D}{(z-1)^2} + \frac{E}{(z+1)(z-1)} \tag{4.100}$$

We will see that the first two terms are not allowed as they will induce third order singularity at infinity. To see this, we can do a transformation $w = 1/z$ (a Mobius transformation) and prove that we will get a third order singularity at $w = 0$. We have;

$$\frac{d}{dz} = \frac{dw}{dz}\frac{d}{dw} = -w^2\frac{d}{dw} \Rightarrow \frac{d^2}{dz^2} = w^2\left(w^2\frac{d^2}{dw^2}\right) = w^4\frac{d^2}{dw^2} + 2w^3\frac{d}{dw} \tag{4.101}$$

So, let's take the first term of the potential in (4.100) only and then, (4.99) becomes;

$$w^4\frac{d^2\psi(w)}{dw^2} + 2w^3\frac{d\psi(w)}{dw} - \frac{A}{1/w+1}\psi(w) = 0 \Rightarrow \frac{d^2\psi(w)}{dw^2} + \frac{2}{w}\frac{d\psi(w)}{dw} - \frac{A}{w^3(w+1)}\psi(w) = 0 \tag{4.102}$$

So, we see that there is a third order pole at $w = 0$ which means that there is a third order pole at $z \to \infty$. The same can be shown for the second term in (4.100) and the remaining terms give second order poles at infinity and thus, they are fine (as they give non trivial constant monodromies). A convenient (and mainstream choice as it makes the variable change that we are going to do later less cluttered) choice for C, D and E is as follows (for example, see [1, 20])

$$V(z) = -\frac{1}{4}\left[\frac{1-\xi_1^2}{(z+1)^2} + \frac{1-\xi_2^2}{(z-1)^2} - \frac{1-\xi_1^2-\xi_2^2+\xi_3^2}{(z+1)(z-1)}\right] \tag{4.103}$$

Since the potential depends on the squares of ξ_i's only, we can (without loss of generality) take ξ_i's to be positive. Now, by doing a variable change, we can transform (4.99) into a hypergeometric equation. The relevant identities and substitutions relevant to hypergeometric equations can be found in many text books on mathematical physics like [32] and [33]. The variable change is as follows;

$$\psi(z) = (z+1)^{(1-\xi_1)/2}(z+1)^{(1-\xi_2)/2}G\left(\frac{z+1}{2}\right) \tag{4.104}$$

where G is a function which will turn out to be a solution of the hypergeometric equation. The reason because of which we had to set the argument of G equal to $(z+1)/2$ is that the solutions of hypergeometric equation have singularities at $0, 1, \infty$ (for example, see [32]) but we want the singularities of $\psi(z)$ to be at $z = \pm 1, \infty$. So, the argument of G is set to shift the singularities from $z = \pm 1$ to $0, 1$. Now, In order to simplify things down, we introduce the following parameters;

$$\xi_1 = 1 - c,\ \xi_2 = c - a - b,\ \xi_3 = a - b \tag{4.105}$$

Then, using (4.104) and (4.105), (4.99) gives (after a slight tedious computation)

$$\mu(1-\mu)\frac{d^2G(\mu)}{d\mu^2} + (c - (a+b+1)\mu)\frac{dG(\mu)}{d\mu} - abG(\mu) = 0 \tag{4.106}$$

where μ is just the independent variable. So, we see that $G(\mu)$ really satisfies the hypergeometric equation while a, b, c can be expressed in terms of ξ_i's by using (4.105). The solutions to the equation (4.106) are denoted as ${}_2F_1(a,b,c;\mu)$ but I will omit the subscripts from now on. There are integral and series expansions for this solution and for interest, here they are [32].

$$F(a,b,c;\mu) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \tag{4.107}$$

where $\Gamma(z)$ is the gamma function and $(a)_n$ is the Pochhammer symbol defined as $(a)_0 = 1$, $(a)_{n+1} = a(a+1)...(a+n)$, $(n \geq 1)$. Nevertheless, we just need to pick two suitable solutions to (4.106) and construct a and a_D out of them. Now, we remember that a doesn't change in the monodromy action around the point at infinity (see (4.60)) and a_D doesn't change in the monodromy action around 1. So, this is what we would expect the solutions that we pick for a and a_D to exhibit as well. Two such solutions are (see [33])

$$G_1(\mu) = (-\mu)^{-a} F\left(a, a+1-c, a+1-b; \frac{1}{\mu}\right) \tag{4.108}$$

$$G_2(\mu) = (1-\mu)^{c-a-b} F\left(c-a, c-b, c+1-a-b; 1-\mu\right) \tag{4.109}$$

Using the series solution in (4.107), it is seen easily that $G_1(\mu)$ is invariant under the monodromy action around the point at infinity and G_2 is invariant under the monodromy action around $\mu = z = 1$.

Now, the last thing that we need to do is to find the values of a, b, c (or equivalently, the values of ξ_1, ξ_2 and ξ_3). To do this, we need to exploit the asymptotic behaviour of a and a_D for large u (and since the independent variable in (4.99) is z, it is equivalent to large z behaviour). Now, for large z, (4.103) becomes (by replacing $z+1, z-1$ by z);

$$V(z) = -\frac{1}{4} \frac{1-\xi_3^2}{z^2} \tag{4.110}$$

and thus, (4.99) becomes an Euler equation i.e.;

$$4z^2 \frac{d^2\psi(z)}{dz^2} + (1-\xi_3^2)\psi(z) = 0 \tag{4.111}$$

The solution to this equation is assumed of the form $\psi(z) = z^l$ and we can easily see that $l = (1 \pm \xi_3)/2$ if $\xi_3 \neq 0$ (as if $\xi_3 = 0$ then we will have only one solution). This will give us the following solution;

$$\psi(z) = z^{(1\pm\xi_3)/2},\ \xi_3 \neq 0\,(\text{ for } z \to \infty) \tag{4.112}$$

In case $\xi_3 = 0$ then we have to employ the method of reduction of order. One solution is then $\sqrt{z}$ and we assume another solution of the form $\phi(z)\sqrt{z}$ (where $\phi(z)$ is an arbitrary differentiable function). Putting $\psi(z) = \phi(z)\sqrt{z}$ in (4.111), we get;

$$z\frac{d^2\phi(z)}{dz^2} + \frac{d\phi(z)}{dz} = 0 \Rightarrow \frac{d\phi(z)}{dz} = \frac{1}{z} \Rightarrow \phi(z) = \ln z \tag{4.113}$$

So, in case of $\xi_3 = 0$, the solutions are $\sqrt{z}$ and $\sqrt{z}\ln z$. Now, using (4.58) and the expression above it for u, we see that in the large u limit, $a \propto \sqrt{u}$ and $a_D \propto \sqrt{u}$ u and thus, the case of $\xi_3 = 0$ should be the correct case to consider.

Now, we can take the limit $z \to 1$ (with $\xi_3 = 0$) and the leading term of (4.103) then becomes;

$$V(z) = -\frac{1}{4} \frac{1-\xi_2^2}{(z-1)^2} \tag{4.114}$$

Now, the equation (4.99) becomes (doing the variable change $y = z - 1 \to z$);

$$\frac{d^2\psi(y)}{dy^2} + \frac{1-\xi_2^2}{4y^2}\psi(y) = 0 \tag{4.115}$$

Now, using the form of a_D around $z = 1$ (or $y = 0$) in (4.74), we see that the second term in (4.115) should vanish in order to attain that solution and thus, $\xi_2 = 1$ (remember that I took ξ_i's to be positive). Now, since $\xi_2 = 1$, it means that the potential doesn't have a double pole at $z = 1$ (see (4.103)). At this point, we need to remind ourselves that due to instanton corrections in the prepotential, the $U(1)_R$ symmetry is broken to $\mathbb{Z}_8$ symmetry on the whole moduli space and thus, as it's subset, there should be a $\mathbb{Z}_2$ symmetry which dictates that if there is no double pole at $z = 1$ then there should not be any double pole at $z = -1$ as well. This implies that $\xi_1 = 1$.

Using these values of ξ_1, ξ_2, ξ_3, we can calculate the values of a, b, c by inverting the equations (4.105) as follows;

$$\begin{aligned} a &= \frac{1}{2}(1 - \xi_1 - \xi_2 + \xi_3) = \frac{1}{2}(1 - 1 - 1 + 0) = -\frac{1}{2} \\ b &= \frac{1}{2}(1 - \xi_1 - \xi_2 - \xi_3) = \frac{1}{2}(1 - 1 - 1 - 0) = -\frac{1}{2} \\ c &= 1 - \xi_1 = 1 - 1 = 0 \end{aligned} \tag{4.116}$$

Using these values in (4.104), we get;

$$\psi(z) = G\left(\frac{z+1}{2}\right) \tag{4.117}$$

and since the solutions for G are given in (4.108) and (4.109), the solutions become (just replace z with u as it is the variable relevant on the moduli space);

$$G_1(u) = i\left(\frac{u+1}{2}\right)^{1/2} F\left(\frac{1}{2}, -\frac{1}{2}, 1; \frac{2}{u+1}\right), \; G_2(u) = -\frac{u-1}{2} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right) \tag{4.118}$$

As mentioned before (below (4.109)), the first solution corresponds to $a(u)$ and the second solution corresponds to $a_D(u)$ (due to their monodromy properties). Now, since (4.99) is linear, I can multiply the solutions by any constant and the resulting function will still be a solution of the original equation. So, the question arises that what multiple of G_1 and G_2 gives the correct asymptotics for a and a_D and using formulas from [33], it is easily seen that the required multiples are;

$$a_D(u) = -iG_2(u) = i\frac{u-1}{2} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right) \tag{4.119}$$

$$a(u) = -2iG_1(u) = \sqrt{2}\,(u+1)^{1/2} F\left(\frac{1}{2}, -\frac{1}{2}, 1; \frac{2}{u+1}\right) \tag{4.120}$$

Now, we have a_D as a function of a through a parametric equations system (u being the parameter) and since $a_D = \mathcal{F}'(a)$, we can integrate with respect to a in order to get $\mathcal{F}(a)$ and thus, we get the low energy effective action which is determined by $\mathcal{F}(a)$ (see (2.135)).

Chapter 5

Conclusion

To conclude this work, we can revise what has been done in this work. We started from the SuperPoincare algebra and developed the massless (and later, massive) supermultiplets. Then the necessary tools were developed (e.g. the concept of chiral superfield and vector superfield) in order to realise the form of the low energy effective $N = 2$ lagrangian. Then, the topics necessary to understand the Olive Montonen duality were developed which included the concept of t'Hooft monopoles, BPS monopoles, the Witten effect and the embedding of non supersymmetric theory in $N = 4$ supersymmetric framework in order to solve the spin problem and the state matching problem lying in front of the proposed Olive Montonen duality. Lastly, the concept of Seiberg Witten duality was developed in the context of low energy $N = 2$ SYM theory with $SU(2)$ gauge group. We saw that the moduli space of the theory is a complex u plane with a varying Higg's expectation value a and a dual expectation value a_D. It was seen that by analyzing the behaviour of a and a_D near the singularities and it's asymptotic limit for large u (i.e. weak coupling), we can determine the value functions a and a_D on the whole u plane.

Following Seiberg and Witten's work, several developments have been made over the years. Many of these developments are in the context of string theory but we won't talk about them here. An obvious generalisation was made by Seiberg and Witten themselves by including matter multiplets (i.e the hypermultiplets) in their analysis [24].

Moreover, there have been works to generalize their work to arbitrary gauge groups. For example, [25] takes a first step in order to generalize Seiberg and Witten's work to $SU(N)$ group. Moreover, [26] calculates low energy effective action for $SU(3)$ gauge group. In addition, [28] does the same work for $SO(2N)$ gauge group. [29] writes down the weak coupling limit for arbitrary gauge groups and writes down the solution for $SO(2r + 1)$ gauge group. Meanwhile, [30] gives the description of the metric on the moduli space for $SU(N)$ gauge group. Lastly, there have been attempts to generalize this work for arbitrary gauge groups and including hypermultiplets. For example, [27] addresses different simple groups (including some exceptional groups) with and without hypermultiplets.

In the end, it should be pointed out that a test for Olive Montonen duality, although difficult to propose (as the test has to be at strong coupling) has been proposed by E. Witten and C. Vafa in [31]. Moreover, test of Seiberg Witten duality using instanton calculus is proposed is [39] which was performed by doing the instanton counting calculations in [40] and the calculations agreed with all literature calculations for low instantons.

Appendix A

Proof of generalized Bogomol'nyi bound

In the presence of electrical fields (keeping $A_0^a = 0$ still), we have the following mass for the dyon;

$$M = \frac{1}{2}\int d^3x[E_i^a E_i^a + B_i^a B_i^a + (D_i\phi^a)^2] \tag{A.1}$$

$$\Rightarrow M = \frac{1}{2}\int d^3x[(E_i^a - cos\,\alpha D_i\phi^a)^2 + (B_i^a - sin\,\alpha D_i\phi^a)^2] + \int d^3x[cos\,\alpha E_i^a D_i\phi^a + sin\alpha B_i^a D_i\phi^a]$$

$$\geq cos\,\alpha\, Q_e + sin\,\alpha\, Q_m \tag{A.2}$$

Now, we can make the bound as tight as possible by differentiating the above expression w.r.t α and setting it to zero. This gives;

$$tan\,\alpha = \frac{Q_m}{Q_e},\ sin\,\alpha = \frac{Q_m}{\sqrt{Q_e^2 + Q_m^2}},\ cos\,\alpha = \frac{Q_m}{\sqrt{Q_e^2 + Q_m^2}} \tag{A.3}$$

This gives us;

$$M \geq v\left[Q_e^2 + Q_m^2\right]^{\frac{1}{2}} \tag{A.4}$$

This is the generalized Bogomol'nyi bound in terms of (Q_e, Q_m).

Appendix B

Poincare Algebra

We know that the angular momentum generator $\hat{\mathbf{M}}$ is the generator of rotations in quantum theory (for example, see [44]). In three dimensions, the angular momentum generator can be defined as a cross product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}} = -i\nabla$ operators but the notion of cross product is special to three dimensions only and can't be generalised to four dimensions. So, we can write another form of the momentum generator which employs the concept of dual tensor M^{ij} for M^i i.e.

$$M^i = \frac{1}{2}\epsilon^{ijk}M^{jk} \tag{B.1}$$

and thus, we have;

$$M^{ij} = x^i\hat{p}^j - x^j\hat{p}^i = -i(x^i\nabla^j - x^j\nabla^i) \tag{B.2}$$

Now, the generalisation to four dimensional spacetime is very simple i.e.;

$$M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{B.3}$$

Now, let's find the commutator of $M^{\mu\nu}$'s. Let f be an arbitrary function on which the generators act. Then we have;

$$[M^{\mu\nu}, M^{\alpha\beta}]f = -[x^\mu\partial^\nu - x^\nu\partial^\mu, x^\alpha\partial^\beta - x^\beta\partial^\alpha]f$$

$$= -x^\mu\partial^\nu(x^\alpha\partial^\beta f - x^\beta\partial^\alpha f) + x^\nu\partial^\mu(x^\alpha\partial^\beta f - x^\beta\partial^\alpha f) + x^\alpha\partial^\beta(x^\mu\partial^\nu f - x^\nu\partial^\mu f) - x^\beta\partial^\alpha(x^\mu\partial^\nu f - x^\nu\partial^\mu f)$$

$$= \left[g^{\alpha\nu}(x^\beta\partial^\mu - x^\mu\partial^\beta) + g^{\nu\beta}(x^\mu\partial^\alpha - x^\alpha\partial^\mu) + g^{\alpha\mu}(x^\nu\partial^\beta - x^\beta\partial^\nu) + g^{\beta\mu}(x^\alpha\partial^\nu - x^\nu\partial^\alpha)\right] f$$

$$= i\left[\left(g^{\mu\beta}M^{\nu\alpha} - g^{\nu\beta}M^{\mu\alpha}\right) - \left(g^{\mu\alpha}M^{\nu\beta} - g^{\nu\alpha}M^{\mu\beta}\right)\right] f = i\left[\left(g^{\mu\beta}M^{\nu\alpha} - (\mu \longleftrightarrow \nu)\right) - (\alpha \longleftrightarrow \beta)\right] f$$

So, we get;

$$[M^{\mu\nu}, M^{\alpha\beta}] = i\left[\left(g^{\mu\beta}M^{\nu\alpha} - (\mu \longleftrightarrow \nu)\right) - (\alpha \longleftrightarrow \beta)\right] \tag{B.4}$$

(B.4) is known as lorentz algebra as it describes rotations in four dimensional spacetime (i.e. lorentz transformations).

For the translations, we consider the following identity;

$$f(x^\mu + a^\mu) = f(x) + a^\mu\partial_\mu f(x) + \mathcal{O}(a^2) = f(x) - ia^\mu\hat{P}_\mu f(x) \text{ where } \hat{P}_\mu = i\partial_\mu \Rightarrow \hat{P}^\mu = i\partial^\mu \tag{B.5}$$

So, we see that the momentum operator $\hat{P}^\mu$ is the generator of translations but since the translations commute, we have (f is again an arbitrary function);

$$[P^\mu, P^\nu] = 0 \tag{B.6}$$

Now, using (B.5) and (B.3), we have;

$$= [P^\mu, M^{\alpha\beta}]f = [i\partial^\mu, i(x^\alpha\partial^\beta 0 x^\beta\partial^\alpha)]f$$

$$= -x^\alpha\partial^\mu\partial^\beta f - g^{\alpha\mu}\partial^\beta f + g^{\mu\beta}\partial^\alpha f + x^\beta\partial^\mu\partial^\alpha f - x^\beta\partial^\alpha\partial^\mu f + x^\alpha\partial^\beta\partial^\mu f$$

$$= (g^{\mu\beta}\partial^\alpha - g^{\alpha\mu}\partial^\beta)f = i(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha)f \tag{B.7}$$

The equations (B.4),(B.6) and (B.7) are collectively known as the **Poincare algebra**.

Bibliography

[1] A. Bilal, "Introduction to supersymmetry," [arXiv:hep-th/0101055 [hep-th]].

[2] M. Bertolini,"Lectures on Supersymmetry", https://people.sissa.it/ bertmat/susycourse.pdf (2020)

[3] N.Seiberg and E.Witten, "Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory", hep-th/9407087, Nucl. Phys. **B426** (1994) 19.

[4] E.Witten and D.I.Olive, "Supersymmetry Algebras That Include Topological Charges," Phys. Lett. B **78**, 97-101 (1978) doi:10.1016/0370-2693(78)90357-X

[5] H. J. W. Muller-Kirsten, Armin Weidemann. "Introduction to Supersymmetry" (2nd edition), World Scientific (2009), 30-32

[6] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory", CRC press (1995)

[7] J.A.Harvey, "Magnetic monopoles, duality and supersymmetry," [arXiv:hep-th/9603086 [hep-th]].

[8] P. A. M. Dirac, "Quantised singularities in the electromagnetic field," Proc. Roy. Soc. Lond. A **133**, no.821, 60-72 (1931) doi:10.1098/rspa.1931.0130

[9] H. Osborn, "Semiclassical Methods for Quantizing Monopole Field Configurations," in Monopoles in Quantum Field Theory, Proceedings of the Monopole Meeting, Trieste Italy 1981, eds. N. S.Craigie, P. Goddard and W. Nahm (World Scientific, Singapore (1982))

[10] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," Phys. Lett. B **72**, 117-120 (1977) doi:10.1016/0370-2693(77)90076-4

[11] C. Callias, " Index theorems on open spaces," Communications in Mathematical Physics **62** (1978) 213

[12] E. Corrigan and D. I. Olive, "Color and Magnetic Monopoles," Nucl. Phys. B **110**, 237-247 (1976) doi:10.1016/0550-3213(76)90525-3

[13] S.R.Coleman, "The Fate of the False Vacuum. 1. Semiclassical Theory," Phys. Rev. D **15**, 2929-2936 (1977) [erratum: Phys. Rev. D **16**, 1248 (1977)] doi:10.1103/PhysRevD.16.1248

[14] B. Zumino, "Supersymmetry and Kahler Manifolds," Phys. Lett. B **87**, 203 (1979) doi:10.1016/0370-2693(79)90964-X

[15] B.de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models," Nucl. Phys. B **245**, 89-117 (1984) doi:10.1016/0550-3213(84)90425-5

[16] A.Strominger, "SPECIAL GEOMETRY," Commun. Math. Phys. **133**, 163-180 (1990) doi:10.1007/BF02096559

[17] L.Alvarez-Gaume and D. Z. Freedman, "Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model," Commun. Math. Phys. **80**, 443 (1981) doi:10.1007/BF01208280

[18] N.Seiberg and E.Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484-550 (1994) doi:10.1016/0550-3213(94)90214-3 [arXiv:hep-th/9408099 [hep-th]].

[19] N. Seiberg, "Supersymmetry and Nonperturbative beta Functions," Phys. Lett. B **206**, 75-80 (1988) doi:10.1016/0370-2693(88)91265-8

[20] A. Bilal, "Duality in N=2 SUSY SU(2) Yang-Mills theory: A Pedagogical introduction to the work of Seiberg and Witten," [arXiv:hep-th/9601007 [hep-th]].

[21] S. Weinberg, "The quantum theory of fields. Vol. 3: Supersymmetry,"

[22] E. Kreyszig. "Advanced Engineering Mathematics" (10th edition), John Wiley publishers

[23] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "EXACT GELL-MANN-LOW FUNCTION IN SUPERSYMMETRIC ELECTRODYNAMICS," Phys. Lett. B **166**, 334 (1986) doi:10.1016/0370-2693(86)90811-7

[24] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484-550 (1994) doi:10.1016/0550-3213(94)90214-3 [arXiv:hep-th/9408099 [hep-th]].

[25] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, "Simple singularities and N=2 supersymmetric Yang-Mills theory," Phys. Lett. B **344**, 169-175 (1995) doi:10.1016/0370-2693(94)01516-F [arXiv:hep-th/9411048 [hep-th]].

[26] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, "Simple singularities and N=2 supersymmetric Yang-Mills theory," Phys. Lett. B **344**, 169-175 (1995) doi:10.1016/0370-2693(94)01516-F [arXiv:hep-th/9411048 [hep-th]].

[27] U. H. Danielsson and B. Sundborg, "Exceptional equivalences in N=2 supersymmetric Yang-Mills theory," Phys. Lett. B **370**, 83-94 (1996) doi:10.1016/0370-2693(95)01566-3 [arXiv:hep-th/9511180 [hep-th]].

[28] A. Brandhuber and K. Landsteiner, "On the monodromies of N=2 supersymmetric Yang-Mills theory with gauge group SO(2n)," Phys. Lett. B **358**, 73-80 (1995) doi:10.1016/0370-2693(95)00986-U [arXiv:hep-th/9507008 [hep-th]].

[29] U. H. Danielsson and B. Sundborg, "The Moduli space and monodromies of N=2 supersymmetric SO(2r+1) Yang-Mills theory," Phys. Lett. B **358**, 273-280 (1995) doi:10.1016/0370-2693(95)01010-N [arXiv:hep-th/9504102 [hep-th]].

[30] P. C. Argyres and A. E. Faraggi, "The vacuum structure and spectrum of N=2 supersymmetric SU(n) gauge theory," Phys. Rev. Lett. **74**, 3931-3934 (1995) doi:10.1103/PhysRevLett.74.3931 [arXiv:hep-th/9411057 [hep-th]].

[31] C. Vafa and E. Witten, "A Strong coupling test of S duality," Nucl. Phys. B **431**, 3-77 (1994) doi:10.1016/0550-3213(94)90097-3 [arXiv:hep-th/9408074 [hep-th]].

[32] J.B. Arfken, H.J. Weber, F.E. Harris; Mathematical methods for physicists, 7th edition, Academic Press, UK, 2013

[33] A. Erdelyi et al; Higher Transcendental Functions, Vol 1, McGraw-Hill, New York, 1953

[34] H. A. Kramers and G. H. Wannier, "Statistics of the two-dimensional ferromagnet. Part 1.," Phys. Rev. **60**, 252-262 (1941) doi:10.1103/PhysRev.60.252

[35] L. Onsager, "Crystal statistics. 1. A Two-dimensional model with an order disorder transition," Phys. Rev. **65**, 117-149 (1944) doi:10.1103/PhysRev.65.117

[36] S. R. Coleman, "The Quantum Sine-Gordon Equation as the Massive Thirring Model," Phys. Rev. D **11**, 2088 (1975) doi:10.1103/PhysRevD.11.2088

[37] S. Mandelstam, "Soliton Operators for the Quantized Sine-Gordon Equation," Phys. Rev. D **11**, 3026 (1975) doi:10.1103/PhysRevD.11.3026

[38] E. J. Weinberg, "Radiative corrections as the origin of spontaneous symmetry breaking," [arXiv:hep-th/0507214 [hep-th]].

[39] A. Lossev, N. Nekrasov and S. L. Shatashvili, "Testing Seiberg-Witten solution," NATO Sci. Ser. C **520**, 359-372 (1999) [arXiv:hep-th/9801061 [hep-th]].

[40] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7**, no.5, 831-864 (2003) doi:10.4310/ATMP.2003.v7.n5.a4 [arXiv:hep-th/0206161 [hep-th]].

[41] D. Tong, "TASI lectures on solitons: Instantons, monopoles, vortices and kinks," [arXiv:hep-th/0509216 [hep-th]].

[42] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The Calculus of many instantons," Phys. Rept. **371**, 231-459 (2002) doi:10.1016/S0370-1573(02)00301-0 [arXiv:hep-th/0206063 [hep-th]].

[43] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," Front. Phys. **58**, 1-548 (1983) [arXiv:hep-th/0108200 [hep-th]].

[44] J. J. Sakurai and J. Napolitano, "Modern Quantum Mechanics," doi:10.1017/9781108587280