Reading Polchinski (Incomplete)

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Contents

Please read this

I am distributing these incomplete notes because some of you asked me to release these incomplete notes. However, please note the following:

1. There are some comments (written in bold) in these notes. Please ignore them.
2. There are some chapters and some parts of the chapters that I still have to write about in these notes. I will keep updating the version of this document on my website. Please keep checking for the latest version (check at least once a month).
3. Some calculations don't agree with Polchinski's book. I still have to figure out if there is a problem in my calculation or if Polchinski's result is wrong (there are a lot of mistakes in Polchinski's books and there is a page of errata for this book (linked below)). When my calculations don't agree with Polchinski's result, I assume Polchinski's result and move on, leaving this work for future.
4. Please let me know if you find any mistakes and/or typos.

Thanks. Let's learn string theory together!!
Links: My Website's page Polchinski's errata page

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2.5.2 bc CFT

The action of the bc CFT is as follows;

$$S = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c \tag{2.5.1}$$

where b and c are anti-commuting primary fields and for the action to be conformally invariant, we need the conformal weights of b and c to be as follows;

$$h_b = \lambda, \;\; h_c = 1 - \lambda$$

The equations of motion of these fields are as follows;

$$\bar{\partial} b = 0 \Rightarrow b = b(z), \; \bar{\partial} c = 0 \Rightarrow c = c(z)$$

and hence, both of the fields are holomorphic. We can now work out the propagators of the theory that have to satisfy the following equation;

$$\bar{\partial}\langle b(z)c(0)\rangle = 2\pi\delta^{(2)}(z) \Rightarrow \bar{\partial}\langle b(z)c(0)\rangle = \bar{\partial}\frac{1}{z} \Rightarrow \langle b(z)c(0)\rangle = \frac{1}{z} \tag{2.5.2}$$

where we used the identity;

$$\bar{\partial}\frac{1}{z} = 2\pi\delta^{(2)}(z)$$

Using (2.5.2), we get the bc OPE;

$$b(z)c(0) \sim \frac{1}{z} \Rightarrow b(z_1)c(z_2) \sim \frac{1}{z_1 - z_2}, \;\; c(z_1)b(z_2) \sim \frac{1}{z_1 - z_2}$$

$$\Rightarrow \partial b(z_1)c(z_2) \sim -\frac{1}{(z_1 - z_2)^2}, \;\; \Rightarrow \partial c(z_1)b(z_2) \sim -\frac{1}{(z_1 - z_2)^2} \tag{2.5.3}$$

We now derive the stress tensor of the theory. It can be done by using Noether's theorem **(do this approach)** but it can be done by another method. Since $T(z)$ should have a conformal weight of 2, there are two possible terms that can appear in $T(z)$ and they are as follows;

$$T(z) = \alpha : b\partial c : (z) + \beta : \partial bc : (z) \tag{2.5.4}$$

where α and β are undetermined coefficients. We can fix α and β by requiring that $b(z)$ and $c(z)$ have the correct weights.

$$T(z)b(0) \sim \alpha : b\partial c : (z)b(0) + \beta : \partial b\, c : (z)b(0) = -\frac{\alpha b(z)}{z^2} + \frac{\beta \partial b(z)}{z} \sim -\frac{\alpha b(0)}{z^2} + \frac{(\beta - \alpha)\partial b(0)}{z}$$

$$T(z)c(0) \sim -\alpha : \partial c\, b : (z)c(0) - \beta : c\partial\, b : (z)c(0) = \frac{\beta c(z)}{z^2} - \frac{\alpha \partial c(z)}{z} \sim \frac{\beta c(0)}{z^2} + \frac{(\beta - \alpha)\partial c(0)}{z}$$

and thus, we see that $\alpha = -\lambda$, $\beta = 1 - \lambda$ and it gives the correct $\beta - \alpha$ value in the residues. Using these values in (2.5.4), we get;

$$T(z) = -\lambda : b\partial c : (z) + (1 - \lambda) : \partial bc : (z) =: \partial bc : (z) - \lambda \partial(: bc : (z)) \tag{2.5.5}$$

Similarly, the TT OPE can be calculated but we will calculate the $1/z^4$ term only to work out the central charge.

$$\frac{1}{z^4} \text{ term} = \lambda^2 : b\partial c : (z) : b\partial c : (0) - \lambda(1 - \lambda) : b\partial c : (z) : \partial b\, c : (0)$$

$$-\lambda(1 - \lambda) : \partial b\, c : (z) : b\partial c : (0) + (1 - \lambda)^2 : \partial b\, c : (z) : \partial b\, c : (0)$$

$$= -\frac{\lambda^2}{z^4} + \frac{2\lambda(1 - \lambda)}{z^4} + \frac{2\lambda(1 - \lambda)}{z^4} - \frac{(1 - \lambda)^2}{z^4} = \frac{1 - 3(2\lambda - 1)^2}{2z^4} \Rightarrow c = 1 - 3(2\lambda - 1)^2 \tag{2.5.6}$$

Talk about ghost number symmetry

2.5.3 $\beta\gamma$ CFT

Same as above but β, γ commute and thus, OPEs change.

2.6 The Virasoro algebra

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2.7 Mode expansions

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2.7.2 bc CFT

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2.7.3 Open Strings

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2.8 Vertex operators

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2.9 More on states and operators

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2.9.1 The OPE

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2.9.2 Virasoro algebra and the highest weight state

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2.9.3 Unitary CFTs

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3 Chapter 3: Polyakov path integral

3.0.1 Sums over worldsheets

He starts by listing the list of string theories and explaining that why can't we have theories with open strings but not closed strings. There are two kinds of arguments;
1) Interactions involving open strings only will eventually give closed strings
2) To disallow interactions that involve joining/splitting of strings and that produce closed strings from open strings will require some non-local constraint.**Expand on this if possible**.

3.1 The Polyakov path integral

The Polyakov action in euclidean metric on the world sheet is given as follows (with the topological term);

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu + \lambda\chi$$

$$= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int_M d^2\sigma\sqrt{g}R + \frac{\lambda}{2\pi}\int_{\partial M} ds\, k \tag{3.1.1}$$

where χ is the Euler's character and k is the geodesic curvature. The expression for χ for genus (number of handles) g and b boundaries is as follows;

$$\chi = 2 - 2g - b$$

Using (3.1), the Polyakov path integral is as follows;

$$\int [dXdg]e^{-S} \tag{3.1.2}$$

(**Include the justification that the Euclidean theory is the same as the Minkowski theory**) We can write (3.1.2) as follows;

$$\int [dXdg]e^{-S} = \int [dXdg]e^{-S_X - \chi\lambda} = \sum_{\text{topologies}} (e^{-\lambda})^\chi \int [dXdg]e^{-S}$$

If we have some stringy process but we add an emitting and reabsorbed open string in this process, we get another boundary, and thus the Euler characteristic increases by 1. So, the term in the path integral gets an additional factor of $e^{-\lambda}$ i.e.

$$e^{-\lambda\chi} \to e^{-\lambda(\chi+1)} = e^{-\lambda\chi}e^{-\lambda}$$

Since the emitting and reabsorbing of the open string should contribute a factor proportional to the square of the open string coupling constant g_o^2, we see that;

$$g_o^2 \sim e^{-\lambda}$$

Similarly, if we have some stringy process but we add an emitting and reabsorbed closed string in this process, we get another handle, and thus the Euler characteristic increases by 2. So, the term in the path integral gets an additional factor of $e^{-2\lambda}$ i.e.

$$e^{-\lambda\chi} \to e^{-\lambda(\chi+2)} = e^{-\lambda\chi}e^{-2\lambda}$$

Since the emitting and reabsorbing of the open string should contribute a factor proportional to the square of the closed string coupling constant g_c^2, we see that;

$$g_c^2 \sim e^{-2\lambda} \Rightarrow g_c \sim e^{-\lambda}$$

So, we have;

$$g_o^2 \sim g_c \sim e^{-\lambda}$$

Please note that these quantities are proportional but not equal to each other. The constants of proportionality will be worked out in chapters 8 and 13. We easily see that closed string sources are boundary loops and open string segments are boundary segments but with endpoints. So, we will need to work out the expression of χ for the endpoints. If there are n_c endpoints in a 2-dimensional surface, then the Euler number is (**derive this**);

$$\chi = 2 - 2g - b - \frac{1}{4}n_c$$

and the path integral weight is (**derive this**);

$$e^{-\lambda\chi + n_c\lambda/4} = e^{-\lambda\tilde{\chi}}$$

where $\tilde{\chi} = \chi - n_c/4$ (**derive this as the λ dependence is not coming right by unitarity.**)

3.2 Gauge fixing

(3.1.2) becomes the following if we divide by the volume of redundant transformations;

$$\int \frac{[dXdg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S[X,g]} \tag{3.2.1}$$

We want the metric to have some functional form (called the fiducial metric $\hat{g}_{ab}$).

3.2.1 The Fadeev Popov determinant

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3.3 The Weyl anomaly

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3.3.1 Calculation of the Weyl anomaly

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3.3.2 Discussion

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3.3.3 Inclusion of boundaries

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3.4 Scattering amplitudes

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3.5 Vertex operators

The vertex operator of a closed string tachyon must have a factor of g_c because an extra string is added. We adopt a normalization to define the closed string tachyon operator as follows;

$$V_0 = 2g_c \int d^2\sigma\, \sqrt{g} : e^{ikX} := 2g_c \int d^2\sigma\ : e^{ikX} : \tag{3.5.1}$$

where in the last step, we used the fact that $g_{ab} = \delta_{ab}$ for the flat worldsheet. We consider the following complex coordinates;

$$z = \sigma^1 + i\sigma^2,\ \ \bar{z} = \sigma^1 - i\sigma^2$$

This gives us the following derivatives;

$$\frac{\partial z}{\partial \sigma^1} = 1,\ \frac{\partial z}{\partial \sigma^2} = i,\ \frac{\partial \bar{z}}{\partial \sigma^1} = 1,\ \frac{\partial \bar{z}}{\partial \sigma^2} = -i$$

which gives the following Jacobian;

$$\begin{vmatrix} \partial z/\partial\sigma^1 & \partial z/\partial\sigma^2 \\ \partial\bar{z}/\partial\sigma^1 & \partial\bar{z}/\partial\sigma^2 \end{vmatrix} = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = 2$$

and thus, we have;

$$d^2\sigma = 2d^2z$$

where we only keep the absolute value. Therefore, we have;

$$V_0 = 2g_c \int d^2\sigma \ : e^{ikX} := g_c \int d^2z \ : e^{ikX} : \tag{3.5.2}$$

This vertex operator must be conformally invariant because it must be Diff and Weyl invariant (**find more justification**). Under the conformal transformation

$$(z, \bar{z}) \to (\tilde{z}, \tilde{\bar{z}})$$

this vertex operator changes as follows;

$$V_0 \to g_c \int d^2z \ : e^{ikX(\tilde{z},\tilde{\bar{z}})} := \int \left(\frac{\partial\tilde{z}}{\partial z}\right)\left(\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}}\right) d^2z \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-\alpha' k^2/4} \left(\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}}\right)^{-\alpha' k^2/4} : e^{ikX(z,z)} :$$

where we used the fact that $: e^{ikX(z,\bar{z})} :$ is a primary field with conformal weights $h = \bar{h} = \alpha' k^2/4$. For V_0 to be conformally invariant, we need the following;

$$1 - \frac{\alpha' k^2}{4} \Longrightarrow k^2 = \frac{4}{\alpha'} \Rightarrow m^2 = -\frac{4}{\alpha'}$$

Thus, the mass of the tachyon state is obtained again. The vertex operator for the gauge boson states is as follows;

$$\frac{2g_c}{\alpha'} \int d^2z : \partial X^\mu \bar{\partial} X^\nu e^{ikX(z,\bar{z})} :$$

The normalization of this vertex operator comes from unitarity (to be explained in chapter 6 **-check this-**). The conformal invariance of this vertex operator implies the following;

$$\frac{2g_c}{\alpha'} \int d^2\tilde{z} : \partial X^\mu \bar{\partial} X^\nu e^{ikX(\tilde{z},\tilde{\bar{z}})} := \frac{2g_c}{\alpha'} \int d^2z : \partial X^\mu \bar{\partial} X^\nu e^{ikX(z,\bar{z})} :$$

$$\Rightarrow \frac{2g_c}{\alpha'} \int \frac{\partial\tilde{z}}{\partial z}\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}} d^2z \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1-\alpha' k^2/4} \left(\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}}\right)^{-1-\alpha' k^2/4} : \partial X^\mu \bar{\partial} X^\nu e^{ikX(z,\bar{z})} := \frac{2g_c}{\alpha'} \int d^2z : \partial X^\mu \bar{\partial} X^\nu e^{ikX(z,\bar{z})} :$$

and thus, we have;

$$\frac{\alpha' k^2}{4} = 0 \Rightarrow k^2 = 0$$

which is expected as the gauge bosons are massless.

3.5.1 Vertex operators in the Polyakov approach

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3.5.2 Off shell amplitudes

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3.5.3 Massless closed string vertex operators

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3.5.4 Open string vertex operators

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3.6 Strings in curved spacetime

3.6.1 Weyl invariance

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3.6.2 Backgrounds

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3.6.3 The spacetime action

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3.6.4 Compactification and CFT

4 Chapter 4: The string spectrum

4.1 Old covariant quantization

The states in the bosonic string include the α^μ_{-n} $(n > 0)$ creation operators and the ghost creation operators which are as follows;

$$b_n \ (n < 0)$$

$$c_n \ (n \le 0)$$

However, some of the α^μ_{-n} oscillators are unphysical because they give negative norm states;

$$||\alpha^0_{-n}|0\rangle||^2 = \langle 0|\alpha^0_n \alpha^0_{-n}|0\rangle = \langle 0|\left[\alpha^0_n, \alpha^0_{-n}\right]|0\rangle = n\eta^{00}\langle 0|0\rangle = n\eta^{00} = -n \ \ (n > 0)$$

(Include the ghost states). So, the actual Hilbert space is smaller than what one would naively guess. An argument that makes it explicit that all the possible states can't be physical states is as follows. If we gauge fix the metric to be $g_{ab}(\sigma)$, then the physical amplitudes should not depend on this gauge choice and if we change the gauge choice to be $g_{ab}(\sigma) + \delta g_{ab}(\sigma)$, then the amplitude should not change. The variation of the amplitude is as follows (**derive this**);

$$\delta\langle f|i\rangle = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g(\sigma)}\delta g_{ab}(\sigma)\langle f|T^{ab}(\sigma)|i\rangle$$

Since this variation should vanish for arbitrary $\delta g_{ab}(\sigma)$ (**Is this arbitrary or the only allowed variation are via Weyl and diff transformations?**), we have;

$$\langle f|T^{ab}(\sigma)|i\rangle = 0 \Rightarrow \langle i|T^{ab}(\sigma)|f\rangle = 0 \ \ \forall a, b$$

So, we see that all possible $|i\rangle$ and $|f\rangle$ states won't satisfy this condition but only a subset of them. So, this condition restricts the number of **physical** states. Therefore, let $|\psi\rangle$ and $|\psi'\rangle$ be physical states. Then, we have;

$$\langle\psi|T_{ab}(\sigma)|\psi'\rangle = 0 \Rightarrow \langle\psi'|T_{ab}(\sigma)|\psi\rangle = 0 \ \ \forall a, b \tag{4.1.1}$$

This condition is equivalent to the following in terms of the stress tensor on the plane;

$$\langle\psi|T(z)|\psi'\rangle = \langle\psi|T_{zz}(z)|\psi'\rangle = \frac{1}{4}\langle\psi|T_{00}(\sigma) - 2iT_{10}(\sigma) - T_{11}(\sigma)|\psi'\rangle = 0$$

$$\langle\psi|\bar{T}(\bar{z})|\psi'\rangle = \langle\psi|\bar{T}_{\bar{z}\bar{z}}(\bar{z})|\psi'\rangle = \frac{1}{4}\langle\psi|T_{00}(\sigma) + 2iT_{10}(\sigma) - T_{11}(\sigma)|\psi'\rangle = 0$$

where we used the following relations between stress tensor components;

$$T(z) = T_{zz}(z) = \frac{1}{4}\left(T_{00}(\sigma) - 2iT_{10}(\sigma) - T_{11}(\sigma)\right) \tag{4.1.2}$$

$$\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}) = \frac{1}{4}\left(T_{00}(\sigma) + 2iT_{10}(\sigma) - T_{11}(\sigma)\right) \tag{4.1.3}$$

Now, using the mode expansion of $T(z)$ and $\bar{T}(\bar{z})$, we have;

$$\langle\psi|T(z)|\psi'\rangle = \sum_{m\in\mathbb{Z}} z^{-m-2}\langle\psi|(L_m + A\delta_{m,0})|\psi'\rangle = 0, \ \ \langle\psi|\bar{T}(\bar{z})|\psi'\rangle = \sum_{m\in\mathbb{Z}} \bar{z}^{-m-2}\langle\psi|(\bar{L}_m + \bar{A}\delta_{m,0})|\psi'\rangle = 0$$

where we have included the normal ordering constants A and $\bar{A}$. Since these equations are true for all z, all the terms of the expansion vanish identically. Thus, we have;

$$\langle\psi|(L_m + A\delta_{m,0})|\psi'\rangle = 0, \ \ \langle\psi|(\bar{L}_m + \bar{A}\delta_{m,0})|\psi'\rangle = 0 \ \ \forall m \in \mathbb{Z}$$

Until now, we have been talking about the full stress tensor. It can be broken down to the stress tensor of the matter CFT T^m_{ab} (which includes X^μ or some X^μ's can be replaced by some other CFT) and the ghost CFT stress tensor T^g_{ab} i.e.;

$$T_{ab} = T^m_{ab} + T^g_{ab}$$

Now, the ad hoc old covariant quantization (OCQ) (which will be shown to be equivalent to the BRST result) condition says that;

$$\langle\psi|T^m(z)|\psi'\rangle = \langle\psi|(L_n^m + A\delta_{n,0})|\psi'\rangle = 0 \ \ \forall n \in \mathbb{Z}$$
$$\langle\psi|\bar{T}^m(\bar{z})|\psi'\rangle = \langle\psi|(\bar{L}_n^m + \bar{A}\delta_{n,0})|\psi'\rangle = 0 \ \ \forall n \in \mathbb{Z}$$

This naively suggests the following condition for a physical state $|\psi\rangle$;

$$(L_n^m + A\delta_{n,0})|\psi'\rangle = 0 \ \ \forall n \in \mathbb{Z}, \quad (\bar{L}_n^m + \bar{A}\delta_{n,0})|\psi'\rangle = 0 \ \ \forall n \in \mathbb{Z} \tag{4.1.4}$$

But this is self-contradictory. To see this, take L_n^m and L_{-n}^m with $n \neq 0$ and $n \neq 1$ and let $|\psi\rangle$ be a physical state. Then, (4.1.4) implies;

$$[L_n^m, L_{-n}^m]|\psi\rangle = L_n^m L_{-n}^m|\psi\rangle - L_{-n}^m L_n^m|\psi\rangle = 0$$

But this can also be calculated using Virasoro algebra;

$$[L_n^m, L_{-n}^m]|\psi\rangle = 2n(L_0^m + A)|\psi\rangle + \frac{D}{12}n(n^2-1)|\psi\rangle = \frac{D}{12}n(n^2-1)|\psi\rangle$$

But this can't be zero as $n \neq 0, 1$ as we assumed above. So, (4.1.4) can't be required to hold for all $n \in \mathbb{Z}$. So, let's require it to hold it for $n \geq 0$ only (**Why not for -1 because it doesn't give contradiction. Or does it?**). So, the correct OCQ condition is as follows;

$$(L_n^m + A\delta_{n,0})|\psi\rangle = 0 \ \ n \leq 0 \tag{4.1.5}$$

It means that all physical states have a conformal weight equal to $-A$. This condition also means that;

$$0 = \langle\psi'|L_n^m|\psi\rangle = \langle\psi'|L_n^m|\psi\rangle^\dagger = \langle\psi|L_n^{\dagger m}|\psi'\rangle = \langle\psi|L_{-n}^m|\psi'\rangle \Rightarrow \langle\psi|L_{-n}^m|\psi'\rangle = 0 \ \ (n > 0)$$

where in the second step, we used the fact that $\langle\psi'|L_n^m|\psi\rangle$ is equal to its conjugate because it is zero. and in the last step, we used the fact that $L_n^{\dagger m} = L_{-n}^m$ which can be proved as follows (there are other ways to do this);

$$L_n^m = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{n-l}.\alpha_l :\Rightarrow L_n^{\dagger m} = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{-l}.\alpha_{-n+l} := \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{-n-l}.\alpha_l := L_{-n}^m \ \ (n \neq 0)$$

where we used the fact the $\alpha_{-l}.\alpha_{-n+l} = \alpha_{-n+l}.\alpha_{-l}$ for $n \neq 0$ and in the last step, we renamed the indices $l \to -l$. We see that the following state;

$$|\chi\rangle = \sum_{n=1}^{\infty} L_{-n}^m|\chi_n\rangle$$

is orthogonal to all physical states for all $|\chi_n\rangle$ as seen below;

$$\langle\psi|\chi\rangle = \sum_{n=1}^{\infty}\langle\psi|L_{-n}^m|\chi_n\rangle = \sum_{n=1}^{\infty}\langle\chi_n|L_{-n}^m|\psi\rangle^\dagger = 0$$

Such a state is called a **spurious** state. If a state is both spurious and physical then it is called a **null** state. The following result is very important;

$$\langle\psi'|\psi\rangle + \langle\psi'|\chi\rangle = \langle\psi'|\psi\rangle$$

where $|\psi\rangle$ and $|\psi'\rangle$ are physical states and $|\chi\rangle$ is a null state. Thus, the dot product of a physical state $|\psi'\rangle$ with any other physical state $|\psi\rangle$ is the same as the inner product of $|\psi'\rangle$ with $|\psi\rangle + |\chi\rangle$. Therefore, as far as physical states are concerned $|\psi\rangle$ and $|\psi\rangle + |\chi\rangle$ are equivalent (**find more justification of equivalence**). So, the space of physical states is the following coset space;

$$\mathcal{H}_{\text{OCQ}} = \mathcal{H}_{\text{physical}}/\mathcal{H}_{\text{null}}$$

Now, we apply (4.1.5) on the lightest open string states. For that, we need the following expressions;

$$L_n^m = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha_{n-l}^\mu \alpha_l^\nu : \eta_{\mu\nu} = \frac{1}{2}\sum_{l=n+1}^{\infty}\alpha_{n-l}^\mu\alpha_l^\nu\eta_{\mu\nu} + \frac{1}{2}\sum_{l=-\infty}^{n}\alpha_l^\nu\alpha_{n-l}^\mu\eta_{\mu\nu} \ \ (n \geq 0) \tag{4.1.6}$$

$$L^m_{-n} = \frac{1}{2}\sum_{l\in\mathbb{Z}} :\alpha^\mu_{-n-l}\alpha^\nu_l: \eta_{\mu\nu} = \frac{1}{2}\sum_{l=1}^{\infty}\alpha^\mu_{-n-l}\alpha^\nu_l\eta_{\mu\nu} + \frac{1}{2}\sum_{l=1}^{n-1}\alpha^\mu_{-n+l}\alpha^\nu_{-l}\eta_{\mu\nu} + \frac{1}{2}\sum_{l=n+1}^{\infty}\alpha^\mu_{-l}\alpha^\nu_{l-n}\eta_{\mu\nu} + \alpha^\nu_0\alpha^\mu_{-n}\eta_{\mu\nu} \quad (n>0) \tag{4.1.7}$$

Now, we have for $n \geq 0$, the lightest state $|0;k\rangle$ gives;

$$L^m_n|0;k\rangle = \frac{1}{2}\sum_{l=n+1}^{\infty}\alpha^\mu_{n-l}\alpha^\nu_l\eta_{\mu\nu}|0;k\rangle + \frac{1}{2}\sum_{l=-\infty}^{n}\alpha^\nu_l\alpha^\mu_{n-l}\eta_{\mu\nu}|0;k\rangle = \frac{1}{2}\sum_{l=-\infty}^{n}\alpha^\nu_l\alpha^\mu_{n-l}\eta_{\mu\nu}|0;k\rangle \tag{4.1.8}$$

This is trivially zero unless $n = 0$. For $n = 0$, we have;

$$L^m_0|0;k\rangle = \frac{1}{2}\sum_{l=-\infty}^{n}\alpha^\nu_l\alpha^\mu_{-l}\eta_{\mu\nu}|0;k\rangle = \frac{1}{2}\alpha^\nu_0\alpha^\mu_0\eta_{\mu\nu}|0;k\rangle = \alpha' k^2|0;k\rangle \tag{4.1.9}$$

where we used the fact that $\alpha^\mu_0 = \sqrt{2\alpha'}\,k^\mu$. So, the only non-trivial constraint on $|0;k\rangle$ is as follows;

$$(L^m_0 + A)|0;k\rangle = (\alpha' k^2 + A)|0;k\rangle = 0 \Rightarrow k^2 = -\frac{A}{\alpha'} \Rightarrow m^2 = \frac{A}{\alpha'}$$

where $m^2 = -k^2$ is the mass of the state. There are no spurious states at this level. The next set of states is as follows;

$$e.\alpha_{-1}|0;k\rangle = e_\beta\alpha^\beta_{-1}|0;k\rangle \tag{4.1.10}$$

We see that from (4.1.6), the first term contributes only if $n = 1$ but that can only happen if $m = 0$. In this case, the second term only contributes for $n = 0$ or -1. So, we get the following;

$$L^m_0 e_\beta\alpha^\beta_{-1}|0;k\rangle = \frac{1}{2}\left(\alpha^\mu_{-1}\alpha^\nu_1\eta_{\mu\nu} + \alpha^\nu_{-1}\alpha^\mu_1\eta_{\mu\nu} + \alpha^\nu_0\alpha^\mu_0\eta_{\mu\nu}\right)e_\beta\alpha^\beta_{-1}|0;k\rangle = \alpha^\mu_{-1}\eta^{\nu\beta}_1\eta_{\mu\nu}e_\beta|0;k\rangle + \alpha' k^2 e_\beta\alpha^\beta_{-1}|0;k\rangle$$

$$= (e.\alpha_{-1})|0;k\rangle + \alpha' k^2(e.\alpha_{-1})|0;k\rangle = \left(1+\alpha' k^2\right)(e.\alpha_{-1})|0;k\rangle$$

where in the second step, we used the α commutation relation and the fact that $\alpha^\mu_1|0;k\rangle = 0$. Therefore, we get;

$$(L^m_0 + A)e_\beta\alpha^\beta_{-1}|0;k\rangle = \left(1 + A + \alpha' k^2\right)(e.\alpha_{-1})|0;k\rangle = 0 \Rightarrow m^2 = \frac{1+A}{\alpha'}$$

In the second term in (4.1.6), the index $n-m$ ranges from 0 to ∞ and the nonzero contributions from this term for (4.1.10) can come when $n - m = 0$ or 1. $n - m = 0$ when $n = m$ and $n - m = 1$ when $n = m+1$. But then we see that the α_n operator from the second term in (4.1.6) will give a nonzero action on the state only if $n = 1$ and. Now, we have already seen the $m = 0$ case above. So, we see that the only nonzero L_m action comes from $m = 1$. In this case, only the $n = 1$ term in the second term in (4.1.6) contributes and thus, we have the following constraint;

$$0 = L_1 e_\beta\alpha^\beta_{-1}|0;k\rangle = \frac{1}{2}\alpha^\nu_1\alpha^\mu_0\eta_{\mu\nu}e_\beta\alpha^\beta_{-1}|0;k\rangle = \frac{1}{2}\sqrt{2\alpha'}\,\eta_{\mu\nu}k^\mu\eta^{\nu\beta}e_\beta|0;k\rangle = e.k|0;k\rangle \Rightarrow e.k = 0$$

We now look for any spurious states. There is only one possible spurious state at level 1 i.e.;

$$L^m_{-1}|0;k\rangle = \sqrt{2\alpha'}\,k^\nu\alpha^\mu_{-1}\eta_{\mu\nu}|0;k\rangle + \frac{1}{2}\sum_{l=2}\alpha^\mu_{-l}\alpha^\nu_{l-1}\eta_{\mu\nu}|0;k\rangle = \sqrt{2\alpha'}\,k.\alpha_{-1}|0;k\rangle$$

where we used (4.1.7) and for $n = 1$, the first and second terms don't contribute while acting on the vacuum. It is easy to see that this state is physical (and thus, null) if $k^2 = 0$ but we will see if this actually happens. There are three possibilities;

- If $A + 1 > 0$, then we have a massive state and if we go to the rest frame, then k^μ can be written as;

$$k^\mu = (m, \underbrace{0, ..., 0}_{d=D-1})$$

  The little group of this momentum is SO(D-1) and there are no null states as $k^2 \neq 0$. It is unknown how to give such theory interactions (**find more about it**). From the closed string analysis, we can prove that this case is ruled out by considering representations of $SO(D-1)$. We will get back to that later.

- If $A + 1 = 0$, then the mass is zero and thus, the spurious state is null. In some frame, we can write k^μ as follows;

$$k^\mu = (k, \underbrace{0, ..., 0}_{D-2}, k)$$

The little group of this momentum is $SO(D-2)$.

- If $A + 1 < 0$, then $m^2 < 0$ and thus, we have a multiplet of tachyons. This case is unacceptable.

Complete the section with $D = 26$ and closed string spectrum

4.1.1 Mnemonic

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4.2 BRST quantization

We now study the procedure of gauge fixing more carefully. Suppose that we have a set of degrees of freedom in our formalism. In the case of a point particle, they are $X^\mu(\tau)$ and the einbein $e(\tau)$ fields but the set of degrees of freedom consists of the values of fields at every point of spacetime and there are such values for every value of the indices that the fields carry (for example, μ index for $X^\mu(\tau)$ field). We will label all degrees of freedom by a single index i. This index includes the indices on the fields and the coordinates (recall the fact we have degrees of freedom at every point of spacetime and when we say that the index includes coordinates as well, we are referring to this fact).
So, all the degrees of freedom (DOF) are referred to as ϕ_i. Now, the finite variation of ϕ_i under the gauge transformation α is $\delta_\alpha \phi_i$. Recall that gauge transformations act with a position-dependent parameter and thus, α labels gauge transformations at every point in spacetime. Since the finite gauge transformation on a particular DOF is referred to as δ_α, the infinitesimal gauge transformation can be written as $\epsilon^\alpha \delta_\alpha$ where ϵ^α is an infinitesimal, real (because it just scaled the finite gauge transformation) parameter that depends on the DOF. Now, if all the gauge transformations close, then the commutator of two gauge transformations give another gauge transformation i.e.;

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta} \delta_\gamma$$

where $f^\gamma_{\alpha\beta}$'s constants. We now describe the gauge fixing condition. The gauge fixing condition is a function of all the DOFs that which set to zero. We have a gauge fixing condition for every point in the spacetime and for every index on the fields. So, there must be an index on the gauge fixing condition as well and this index also contains the coordinates and field indices. So, the gauge fixing condition is written as follows;

$$F^A(\phi) = 0$$

where A is the index described above. Now, if the original action of the fields (which is gauge invariant) is denoted as S_1, we impose gauge fixing conditions for all A and to do this, we will need Lagrange multipliers for every A. We denote these Lagrange multipliers as $-iB_A$. Adding these Lagrange multipliers, the action (i.e. the quantity to be extremized) becomes;

$$S_1 - iB_A F^A(\phi) = S_1 + S_2 \text{ where } S_2 = -iB_A F^A(\phi)$$

Please note that the summation on A includes an integration on all spacetime and thus, the Lagrange multiplier term can be thought of as an action. To make an analogy, in the Fadeev Popov procedure for the string, ϕ_i's are $X^\mu(\sigma)$ and $g_{ab}(\sigma)$ while B_A culminate into ζ. When we do the Fadeev Popov procedure, we get (**fill in the details**);

$$\int \frac{[d\phi_i]}{V_{\text{gauge}}} \exp(-S_1) = \int [d\phi_i dB_A db_A dc^\alpha] \exp(-S_1 - S_2 - S_3)$$

where S_3 is the Fadeev Popov action;

$$S_3 = b_A c^\alpha \delta_\alpha F^A(\phi)$$

Note that its form matches the Fadeev action that we got for the string;

$$\int d^2\sigma \sqrt{\hat{g}(\sigma)} b_{ab} (\hat{P}_1 c)^{ab}$$

Recall that $\hat{P}_1$ comes from the variation of the metric and the other term in the variation of the metric has zero contraction with b_{ab} as b_{ab} is traceless and thus, the contraction with this other term doesn't appear in the action above. So, the total action becomes;

$$S_1 + S_2 + S_3 = S_1 - iB_A F^A(\phi) + b_A c^\alpha \delta_\alpha F^A(\phi) \tag{4.2.1}$$

We now prove a couple of facts. Firstly, we show that (4.2.1) is invariant under the following transformation called the BRST transformation (the subscript B refers to BRST transformation);

$$\begin{aligned} &\delta_B \phi_i = -i\epsilon c^\alpha \delta_\alpha \phi_i, \quad \delta_B B_A = 0 \\ &\delta_B b_A = \epsilon B_A, \quad \delta_B c^\alpha = \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma \end{aligned} \tag{4.2.2}$$

Since the variation of anti-commuting variables (b_A and c^α) contains commuting variables e.g. $\delta_B b_A = \epsilon B_A$, the parameter ϵ should be anti-commuting with b_A and c^α.
If we want to have ghost number symmetry (i.e. ghost numbers which are conserved under (4.2.2)), then we need to assign some ghost numbers to the anticommuting variables i.e. b_A, ϵ and c^α. The ghost numbers of the commuting fields will be zero. From the variation of ϕ_i in (4.2.2), we see that c^α and ϵ must have ghost numbers which are negatives of each other to keep the ghost number of ϕ_i at zero. Moreover, from the variation of b_A, we see that the ghost number of b_A and ϵ should be equal. So, we assign ghost number +1 to c^α and −1 to b_A and ϵ.
We now prove that the action (4.2.1) is invariant under (4.2.2). The variation of the actions are as follows;

$$S_1 \to S_1 \tag{4.2.3}$$

because S_1 is already gauge invariant and $\delta_B \phi_i$ is nothing but a gauge transformation with the parameter ϵ^α being $-i\epsilon c^\alpha$. Moreover, we have;

$$\begin{aligned} S_2 \to S_2 - iB_A \delta_B F^A(\phi) &= S_2 - iB_A \delta_B \phi_i \partial_i F^A(\phi) = S_2 - iB_A \left(-i\epsilon c^\alpha \delta_\alpha \phi_i\right) \partial_i F^A(\phi) \\ &= S_2 - \epsilon B_A c^\alpha \delta_\alpha \phi_i \partial_i F^A(\phi) = S_2 - \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) \end{aligned} \tag{4.2.4}$$

Lastly, we have;

$$\begin{aligned} S_3 \to &(b_A + \epsilon B_A)\left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\left(\delta_\alpha F^A(\phi) + \delta_B \phi_B \partial_i \delta_\alpha F^A(\phi)\right) \\ &= (b_A + \epsilon B_A)\left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\left(\delta_\alpha F^A(\phi) - i\epsilon c^\beta \delta_\beta \phi_i \partial_i \delta_\alpha F^A(\phi)\right) \\ &= (b_A + \epsilon B_A)\left(c^\alpha + \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\left(\delta_\alpha F^A(\phi) - i\epsilon c^\beta \delta_\beta \delta_\alpha F^A(\phi)\right) \\ &= S_3 - \frac{i}{2}\epsilon b_A f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha F^A(\phi) + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) - i\epsilon b_A c^\alpha c^\beta \delta_\beta \delta_\alpha F^A(\phi) \end{aligned}$$

where all the ϵ^2 terms vanish as ϵ is anticommuting. Now, using the following fact;

$$c^\alpha c^\beta \delta_\beta \delta_\alpha = \frac{1}{2}\left(c^\alpha c^\beta \delta_\beta \delta_\alpha - c^\beta c^\alpha \delta_\beta \delta_\alpha\right) = -\frac{1}{2}\left(c^\alpha c^\beta \delta_\alpha \delta_\beta - c^\alpha c^\beta \delta_\beta \delta_\alpha\right) = -\frac{1}{2} c^\alpha c^\beta \left[\delta_\alpha, \delta_\beta\right] = -\frac{1}{2} c^\alpha c^\beta f^\gamma_{\alpha\beta} \delta_\gamma$$

we get;

$$S_3 \to S_3 - \frac{i}{2}\epsilon b_A f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha F^A(\phi) + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + \frac{i}{2}\epsilon b_A c^\alpha c^\beta f^\gamma_{\alpha\beta} \delta_\gamma F^A(\phi) = S_3 + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) \tag{4.2.5}$$

Hence, we see from (4.2.4) and (4.2.5) that $S_2 + S_3$ is invariant. Also using (4.2.3), we see that $S_1 + S_2 + S_3$ is invariant under (4.2.2) and thus, it is BRST invariant.
We now see how $b_A F^A(\phi)$ changes under (4.2.2). It changes as follows;

$$\begin{aligned} b_A F^A(\phi) \to (b_A + \epsilon B_A)\left(F_A(\phi) + \delta_B \phi_i \partial_i F_A(\phi)\right) = (b_A + \epsilon B_A)\left(F_A(\phi) - i\epsilon c^\alpha \delta_\alpha \phi_i \partial_i F_A(\phi)\right) \\ = (b_A + \epsilon B_A)\left(F_A(\phi) - i\epsilon c^\alpha \delta_\alpha F_A(\phi)\right) = b_A F^A + \underbrace{i\epsilon\left(b_A c^\alpha \delta_\alpha F_A(\phi) - iB_A F^A(\phi)\right)}_{\delta_B(b_A F^A)} = b_A F^A + \underbrace{i\epsilon\left(S_2 + S_3\right)}_{\delta_B(b_A F^A)} \end{aligned}$$

So, we have;

$$\delta_B(b_A F^A) = i\epsilon\,(S_2 + S_3) \tag{4.2.6}$$

Now, we change the gauge fixing condition by a finite variation $\delta F^A(\phi)$. The infinitesimal variation will be controlled by a parameter ϵ. So, the infinitesimal variation in $F^A(\phi)$ is $\epsilon\delta F^A(\phi)$. Now, using the path integral formalism the expression of the amplitude $\langle f|i\rangle$ is;

$$\langle f|i\rangle = \int [d\phi_i]\, e^{-S_1} \xrightarrow{\text{gauge fixing}} \int [d\phi_i db_A dB_A dc^\alpha]\, e^{-(S_1+S_2+S_3)}$$

where the integrations of b_A, B_A and c^α come due to gauge fixing and Fadeev Popov gauge determinant. Now, the variation of this amplitude gives;

$$\langle f|i\rangle \to \int [d\phi_i db_A dB_A dc^\alpha]\, e^{-(S_1+S_2+S_3)} \left(1 - \epsilon\delta(S_2+S_3) + \mathcal{O}(\epsilon^2)\right) = \langle f|i\rangle - \epsilon\langle f|\delta(S_2+S_3)|i\rangle + \mathcal{O}(\epsilon^2)$$

where we used the fact that S_1 doesn't depend on $F^A(\phi)$ and thus, $\delta S_1 = 0$ for the variation in $F^A(\phi)$. Using (4.2.6), we have;

$$\epsilon\delta\langle f|i\rangle = -\epsilon\langle f|\delta(S_2+S_3)|i\rangle = i\langle f|\delta_B(b_A\delta F^A)|i\rangle$$

Now, we know that the variation of ϕ_i under a symmetry can be written as a quantity that is proportional to the commutator of the current (which is conserved under this symmetry) with ϕ_i but this is true if the parameter of that symmetry is a commuting variable. If the parameter of that symmetry is an anticommuting variable, then we have anticommutator instead of a commutator (**write more about this**). For the BRST symmetry, the parameter is anticommuting and thus, we have;

$$\epsilon\delta\langle f|i\rangle = i\langle f|\delta_B(b_A\delta F^A)|i\rangle \propto \epsilon\langle f|\{Q_B, b^A\delta F_A\}|i\rangle \tag{4.2.7}$$

where Q_B is the BRST charge. In order for this variation of amplitude to vanish, we must have;

$$\langle f|\{Q_B, b^A\delta F_A\}|i\rangle = 0 \quad \text{for arbitrary } \delta F^A(\phi) \tag{4.2.8}$$

So, physical states must satisfy the following for arbitrary δF^A;

$$\langle\psi|\{Q_B, b^A\delta F_A\}|\psi'\rangle = 0 \Rightarrow \langle\psi|Q_B b^A\delta F_A|\psi'\rangle - \psi|b^A\delta F_A Q_B|\psi'\rangle = 0$$

Now, this constraint is satisfied if for all physical states, $Q_B|\psi\rangle = 0$ and if we assume that $Q_B^\dagger = Q_B$, then it also implies that $\langle\psi|Q_B = 0$. $Q_B^\dagger = Q_B$ because the gauge parameter ϵ^α was real (**elaborate more on this**). Since the variation in F^A can be arbitrary, we can choose it to be;

$$\delta F^A = -iB_B M^{AB}$$

for some arbitrary constant matrix. We immediately see that δ_α variation for this variation of F^A is zero and thus, S_3 is invariant. However, S_2 changes as follows;

$$S_2 \to -iB_A\left(F^A - iB_B M^{AB}\right) = S_2 - B_A B_B M^{AB}$$

Moreover, S_1 doesn't change. So, effectively we are adding one term to the action i.e. ;

$$S_1 + S_2 + S_3 \to S_1 + S_2 + S_3 - B_A B_B M^{AB}$$

and thus, the B_A integration now gives a gaussian instead of a delta function.
Since the action in (4.2.1) has BRST invariance, the BRST charge must be conserved if we change the gauge choices (i.e. F^A). So, the BRST charge must commute with the variation of the Hamiltonian of the theory. We are requiring it for the variation because we are assuming that the starting action does have BRST symmetry and this symmetry should be preserved if we change the action (by changing the gauge choice). For this to happen, Q_B must commute with the full hamiltonian but it already commutes with the original hamiltonian and thus, we only require that it commutes with the variation of the hamiltonian. Now, for the amplitude $\langle f|i\rangle$, we actually have a factor of e^{-iHt} when we are considering amplitudes at different times. So, we have;

$$\langle f|e^{-iHt}|i\rangle \to \langle f|e^{-i(H+\delta H)t}|i\rangle = \langle f\left|e^{-i(H)t}\left(1 - it\delta H\right)\right|i\rangle + \mathcal{O}((\delta H)^2)$$

$$= \langle f\left|e^{-i(H)t}\right|i\rangle + -it\langle f\left|e^{-i(H)t}\delta H\right|i\rangle + \mathcal{O}((\delta H)^2)$$

Now, if we compare this result to (4.2.7) (with the understanding that the factor e^{-iHT} will come there if the final and initial states are at different times), we see that δH should be proportional to (**is this the right argument?**);

$$\{Q_B, b^A\delta F_A\}$$

and thus, the commutator of Q_B to vanish with δH implies;

$$\left[Q_B, \{Q_B, b_A\delta F^A\}\right] = 0$$

This implies;

$$Q_B^2 b_A\delta F^A - b_A\delta F^A Q_B^2 = \left[Q_B^2, b_A\delta F^A\right] = 0 \tag{4.2.9}$$

for arbitrary δF^A. Before exploring its consequence, look at (4.2.7) and since the ghost number of b_A is -1, and the ghost number of the variation of $\langle f|i\rangle$ is zero, the ghost number of Q_A should be +1 (the ghost number of $F^A(\phi)$ is also zero as it depends on ϕ_i's). So, the ghost number of Q_B^2 is +2, and to satisfy (4.2.9) we can't take it to be a non-zero constant because it will get changed by the ghost number symmetry (**Is this argument correct?**). Hence, we have to set $Q_B^2 = 0$ to satisfy (4.2.9). Hence, we have;

$$Q_B^2 = 0 \tag{4.2.10}$$

(The issue of having linear terms in twice BRST, the BV formalism and cohomology thing).

4.2.1 Point particle example

In the point particle case, the coordinate is τ and the fields are $X^\mu(\tau)$ and $e(\tau)$ so the index α in the gauge transformation δ_α in BRST formalism should correspond to τs. We need to choose a basis for gauge transformations. A way to choose this is to realize gauge transformation at a point τ_1 should not change any τ but τ_1 itself. So, a basis δ_{τ_1} is such that;

$$\delta_{\tau_1}\tau = \delta(\tau - \tau_1)$$

A general transformation of τ will be then;

$$\delta\tau = \epsilon^{\tau_1}\delta_{\tau_1}\tau = \epsilon(\tau_1)\delta(\tau-\tau_1) = \epsilon(\tau) \Rightarrow \tau \to \tau + \eta(\tau)$$

We know that under this transformation we get (from the analysis of point particle action before);

$$\delta X^\mu(\tau) = -\eta(\tau)\partial_\tau X^\mu(\tau) = -\epsilon^{\tau_1}\delta_{\tau_1}\partial_\tau X^\mu(\tau) = -\epsilon^{\tau_1}\delta(\tau-\tau_1)\partial_{\tau_1}X^\mu(\tau_1) \Rightarrow \delta_{\tau_1}X^\mu = -\delta(\tau-\tau_1)\partial_{\tau_1}X^\mu(\tau_1)$$

$$\delta e(\tau) = -\partial_\tau\left(\epsilon(\tau)e(\tau)\right) = -\epsilon^{\tau_1}\partial_\tau\left(\delta_{\tau_1}e(\tau)\right) = -\epsilon^{\tau_1}\partial_{\tau_1}\left(\delta(\tau-\tau_1)e(\tau_1)\right) \Rightarrow \delta_{\tau_1}e(\tau) = -\partial_{\tau_1}\left(\delta(\tau-\tau_1)e(\tau_1)\right)$$

To find the structure constants of the BRST algebra, we need to compute $[\delta_{\tau_1}, \delta_{\tau_2}]$. We compute this commutator acting on $X^\mu(\tau)$. It turns out to be as follows;

$$[\delta_{\tau_1}, \delta_{\tau_2}]X^\mu(\tau) = \delta_{\tau_1}\delta_{\tau_2}X^\mu(\tau) - (1 \leftrightarrow 2)$$

Now, we have;

$$\delta_{\tau_1}\delta_{\tau_2}X^\mu(\tau) = \delta_1\left(\delta(\tau-\tau_2)\partial_\tau X^\mu(\tau)\right) = \delta(\tau-\tau_2)\partial_\tau\left(\delta_1 X^\mu(\tau)\right) = \delta(\tau-\tau_2)\partial_\tau\left(\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau)\right)$$

$$= \delta(\tau-\tau_2)\delta(\tau-\tau_1)\partial_\tau^2 X^\mu(\tau) + \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau)$$

The first term gets canceled with the $1 \leftrightarrow 2$ term. Therefore, we get;

$$[\delta_{\tau_1}, \delta_{\tau_2}]X^\mu(\tau) = \left[\delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1) - \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\right]\partial_\tau X^\mu(\tau)$$

For this commutator to have the form $f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X^\mu$ (the summation over τ_3 is actually an integral over τ_3), we can rewrite this commutator as follows;

$$[\delta_{\tau_1}, \delta_{\tau_2}]X^\mu(\tau) = \left[\delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1) - \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\right]\partial_\tau X^\mu(\tau)$$

$$= \int d\tau_3 \left[\delta(\tau-\tau_3)\delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1) - \delta(\tau-\tau_3)\delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2)\right]\partial_\tau X^\mu(\tau)$$

$$= -\int d\tau_3 \left[\delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1) - \delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2)\right]\delta_{\tau_3}X^\mu(\tau) = \int d\tau_3 f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X^\mu(\tau)$$

So, we see that;

$$f^{\tau_3}_{\tau_1\tau_2} = \delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2) - \delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)$$

Now, we can calculate the BRST transformations as follows;

$$\delta_B X^\mu(\tau) = -\int d\tau_1 i\epsilon c^{\tau_1}\delta_{\tau_1}X^\mu(\tau) = i\epsilon\int d\tau_1 c^{\tau_1}\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau) = i\epsilon c(\tau)\partial_\tau X^\mu(\tau) = i\epsilon c(\tau)\dot{X}^\mu(\tau) \quad (4.2.11)$$

$$\delta_B e(\tau) = -i\epsilon\int d\tau_1 c^{\tau_1}\delta_{\tau_1}e(\tau) = i\epsilon\int d\tau_1 c^{\tau_1}\partial_\tau\left(\delta(\tau-\tau_1)e(\tau)\right)$$

$$= i\epsilon\partial_\tau\int d\tau_1 c(\tau_1)\delta(\tau-\tau_1)e(\tau) = i\epsilon\partial_\tau\left(c(\tau)e(\tau)\right) = i\epsilon\dot{(ce)} \quad (4.2.12)$$

$$\delta_B B(\tau) = 0 \quad (4.2.13)$$

$$\delta_B b(\tau) = \epsilon B(\tau) \quad (4.2.14)$$

$$\delta_B c^{\tau_1} = \delta_B c(\tau_1) = \frac{i}{2}\epsilon f^{\tau_1}_{\tau_2\tau_3}c^{\tau_2}c^{\tau_3} = \frac{i}{2}\epsilon\int d\tau_2\, d\tau_3\left[\delta(\tau_1-\tau_2)\partial_{\tau_1}\delta(\tau_1-\tau_3) - \delta(\tau_1-\tau_3)\partial_\tau\delta(\tau_1-\tau_2)\right]c(\tau_2)c(\tau_3)$$

$$= \frac{i\epsilon}{2}\left[\int d\tau_3\partial_{\tau_1}\delta(\tau_1-\tau_3)c(\tau_1)c(\tau_3) - \int d\tau_2\partial_{\tau_1}\delta(\tau_1-\tau_2)c(\tau_2)c(\tau_1)\right] = -i\epsilon\int d\tau_3\partial_{\tau_1}\delta(\tau_1-\tau_3)c(\tau_3)c(\tau_1)$$

$$= -i\epsilon\partial_{\tau_1}\left[\int d\tau_3\delta(\tau_1-\tau_3)c(\tau_3)\right]c(\tau_1) = i\epsilon c(\tau_1)\partial_{\tau_1}c(\tau_1) \Rightarrow \delta_B c(\tau) = i\epsilon c(\tau)\partial_\tau c(\tau) = i\epsilon c(\tau)\dot{c}(\tau) \quad (4.2.15)$$

where we used the anticommutativity of c's repeatedly and in the second line, we renamed the τ_2 integration variable to τ_3. For the point particle, the gauge fixing condition is $e(\tau) = 1$ and thus, $F^\tau = 1 - e(\tau)$ the three actions are as follows;

$$S_1 = \frac{1}{2}\int d\tau\left(e^{-1}\dot{X}^\mu\dot{X}_\mu + em^2\right) \quad (4.2.16)$$

$$S_2 = -i\int d\tau B_\tau F^\tau = -i\int d\tau B(\tau)(1-e(\tau)) -= i\int d\tau B(\tau)(e(\tau)-1) \quad (4.2.17)$$

$$S_3 = \int d\tau\int d\tau_1 b_\tau c^{\tau_1}\delta_{\tau_1}F^\tau = \int d\tau\int d\tau_1 b(\tau)c(\tau_1)\delta_{\tau_1}(1-e(\tau)) = \int d\tau\int d\tau_1 b(\tau)c(\tau_1)\partial_\tau\left(\delta(\tau-\tau_1)e(\tau)\right)$$

$$= \int d\tau b(\tau)\partial_\tau\left(\int d\tau_1 c(\tau_1)\delta(\tau-\tau_1)\right)e(\tau) = \int d\tau b(\tau)\dot{c}(\tau)e(\tau) \quad (4.2.18)$$

The total action is thus,

$$S = \int d\tau\left(\frac{1}{2}e^{-1}\dot{X}^\mu\dot{X}_\mu + \frac{e}{2}m^2 - iB(e-1) + b\dot{c}e\right) = \int d\tau\left(\frac{1}{2}e^{-1}\dot{X}^\mu\dot{X}_\mu + \frac{e}{2}m^2 - iB(e-1) - e\dot{b}c\right)$$

(**Justify the last step. By using** $e = 1$ **constraint?**). The EOM for B and e are as follows;

$$B \text{ EOM} : e = 1$$

$$e \text{ EOM} : -\frac{1}{2}e^{-2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - iB - \dot{b}c = 0 \Rightarrow B = -i\left[\frac{1}{2}e^{-2}\dot{X}^\mu\dot{X}_\mu - \frac{1}{2}m^2 + \dot{b}c\right] = i\left[-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right]$$

where in the last step, we set $e = 1$ from B's EOM. Using these equations of motion, we get;

$$S = \int d\tau\left(\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right) \quad (4.2.19)$$

The BRST transformation in (4.2.14) thus changes to;

$$\delta_B b = i\epsilon\left[-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right] \quad (4.2.20)$$

with (4.2.11) and (4.2.15) remaining unchanged. These transformations leave (4.2.19) invariant as we can check. The variation of the X^μ term is;

$$\frac{1}{2}\dot{X}^\mu\dot{X}_\mu \to \frac{1}{2}\left(\partial_\tau\left(X^\mu + i\epsilon c\dot{X}^\mu\right)\partial_\tau\left(X_\mu + i\epsilon c\dot{X}_\mu\right)\right) = \frac{1}{2}\left(\dot{X}^\mu + i\epsilon\dot{c}\dot{X}^\mu + i\epsilon c\ddot{X}^\mu\right)\left(\dot{X}_\mu + i\epsilon\dot{c}\dot{X}_\mu + i\epsilon c\ddot{X}_\mu\right)$$

$$= \frac{1}{2}\dot{X}^\mu\dot{X}_\mu + i\epsilon\dot{c}\dot{X}^\mu\dot{X}_\mu + i\epsilon c\ddot{X}^\mu\dot{X}_\mu$$

The mass term is invariant. The $-\dot{b}c$ term changes as follows;

$$-\dot{b}c \to -\partial_\tau\left(b - i\epsilon\left(\frac{1}{2}\dot{X}^\mu\dot{X}_\mu - \frac{1}{2}m^2 + \dot{b}c\right)\right)(c + i\epsilon c\dot{c}) = \left(-\dot{b} + i\epsilon\dot{X}^\mu\ddot{X}_\mu + i\epsilon\dot{b}c + i\epsilon\dot{b}\dot{c}\right)(c + i\epsilon c\dot{c}) = -\dot{b}c + i\epsilon c\dot{X}^\mu\ddot{X}_\mu$$

(**The total action is not coming as invariant. Find flaw. Also, check the nilpotency of Q_B^2 and the special case of** b). We know these basic facts about the point particle and the ghost system;

$$[X^\mu, p^\nu] = i\eta^{\mu\nu}, ;\ \{b, c\} = 1,\ \ H = \frac{1}{2}\left(p^2 + m^2\right)$$

where because of the Euclidean signature, $\dot{X}^\mu$ is replaced by $-i\dot{X}^\mu$ and thus

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu_l} = \frac{\partial L}{-i\partial \dot{X}^\mu_e} = \frac{\partial L}{-i\partial \dot{X}^\mu} = i\frac{\partial L}{\partial \dot{X}^\mu} = i\dot{X}_\mu$$

where $\dot{X}^\mu_l$ and $\dot{X}^\mu_e$ are the derivatives of X^μ with respect to Lorentzian and Euclidean time i.e. τ_l and τ_e which are related as $\tau_l = i\tau_e$. Moreover, τ_e is just τ. Now, we can construct Noether's current for the BRST symmetry using Noether's theorem as follows (L is the lagrangian);

$$Q_B = \frac{\partial L}{-i\partial \dot{X}^\mu}\Delta X^\mu + \frac{\partial L}{-i\partial \dot{b}}\Delta b + \frac{\partial L}{-i\partial \dot{c}}\Delta c = i\dot{X}^\mu\left(ic\dot{X}^\mu\right) + ic\left[i\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right)\right]$$

$$= \frac{c}{2}\left(-\dot{X}^\mu\dot{X}_\mu + m^2\right) = \frac{c}{2}\left(p^2 + m^2\right) = cH$$

where we recall that the derivative w.r.t $\dot{b}$ anti commutes with $\dot{b}$. Now, the two-level system generated by ghosts is as follows;

$$|k;\uparrow\rangle,\ \ |k;\downarrow\rangle$$

where the following is true;

$$p^\mu|k;\uparrow\rangle = p^\mu|k;\downarrow\rangle = k^\mu$$
$$b|k;\uparrow\rangle = |k;\downarrow\rangle,\ \ b|k;\downarrow\rangle = 0$$
$$c|k;\uparrow\rangle = 0,\ \ c|k;\downarrow\rangle = |k;\uparrow\rangle \tag{4.2.21}$$

This implies that;

$$Q_B|k;\downarrow\rangle = \frac{c}{2}(p^2 + m^2)|k;\downarrow\rangle = \frac{1}{2}(k^2 + m^2)|k;\uparrow\rangle,\ \ Q_B|k;\uparrow\rangle = 0 \quad \text{as } c|k;\uparrow\rangle = 0$$

Therefore the closed states (i.e. the states annihilated by Q_B) are all $|k;\uparrow\rangle$ states and $|k;\downarrow\rangle$ states but only if $k^2 + m^2$. So, the mass shell condition is satisfied by the closed states. As seen from above manipulation, $|k;\downarrow\rangle$ are proportional to $Q_B|k;\uparrow\rangle$ but are non-vanishing only for $k^2 + m^2 \neq 0$. So, these are the exact states. Therefore the closed states that are not exact (i.e. the physical states) are;

$$|k;\downarrow\rangle,\ \ |k;\uparrow\rangle \quad \text{for } k^2 + m^2 = 0$$

We now see that $|k;\uparrow\rangle$ states don't make sense kinematically. Since the amplitudes of the exact states must vanish identically, we see that the amplitudes involving $|k;\uparrow\rangle$ states must vanish if $k^2 + m^2 \neq 0$ because then $|k;\uparrow\rangle$ is an exact state but the amplitudes may not vanish for $|k;\uparrow\rangle$ is $k^2 + m^2 = 0$. So, amplitudes must have $\delta(k^2 + m^2)$ in their expression and this is very peculiar for amplitudes (that aren't in two dimensions) **(Exapnd more on this)**. So, $|k;\uparrow\rangle$ terms don't comprise the physical spectrum and only $|k;\downarrow\rangle$ come in the physical spectrum. This projection can also be obtained by imposing the constraint that;

$$b|\psi\rangle = 0$$

and then, only $|k;\downarrow\rangle$ satisfies this constraint.

4.3 BRST quantization of the string

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4.3.1 BRST cohomology of the string

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4.4 The no ghost theorem

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4.4.1 Proof

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4.4.2 BRST-OCQ equivalence

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5 Chapter 5: The string S-matrix

5.1 The circle and the torus

On the torus, we have a metric modulus τ which appears in the metric as follows;

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$$

It appears here because we choose to keep the periodicities of $\sigma^{1,2}$ the same after the reparametrization that keeps the form of the following metric invariant. These parameterizations are $SL(2,\mathbb{Z})/Z_2$ parametrizations which are given as follows;

$$\begin{pmatrix} \sigma^{1'} \\ \sigma^{2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}, \quad ad - bc = 1 \quad a,b,c,d \in \mathbb{Z}$$

Apart from this, the form of the metric and periodicities are left invariant by rigid translations i.e.

$$\sigma^a \to \sigma^a + \xi^a$$

Such a group of symmetries that is left invariant by the choice of metric is called the conformal killing group (CKG). Thus, we will need to fix the position of a vertex operator to fix this symmetry.

5.2 Moduli and Riemann surfaces

If we want to find the metric deformations which are not available from the diffeomorphism and weyl transformations (diff × Weyl), then we need to find the variations δg_{ab} which are orthogonal to the metric variations that we get from diff × Weyl i.e.;

$$-2(P_1\delta\sigma)_{ab} + (2\delta\omega - \nabla.\delta\sigma)g_{ab} \tag{5.2.1}$$

where P_1 operator is defined as follows by its action on the variation $\delta\sigma^a$;

$$(P_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - g_{ab}\nabla_c\delta\sigma^c)$$

Now, the metric variation δg_{ab} orthogonal to (5.2.1) is defined as follows;

$$\int d^2\sigma\sqrt{g}\,\delta g_{ab}(-2(P_1\delta\sigma)^{ab} + (2\delta\omega - \nabla.\delta\sigma)g^{ab}) = 0$$

$$\Rightarrow \int d^2\sigma\sqrt{g}(-2\delta g_{ab}(P_1\delta\sigma)^{ab} + g^{ab}\delta g_{ab}(2\delta\omega - \nabla.\delta\sigma)) = 0$$

Now, we see that;

$$\delta g_{ab}(P_1\delta\sigma)^{ab} = \delta g_{ab}\nabla^a\delta\sigma^b - \frac{g^{ab}\delta g_{ab}}{2}\nabla.\delta\sigma = \nabla^a(\delta g_{ab}\delta\sigma^b) - (\nabla^a\delta g_{ab})\delta\sigma^b - \frac{g^{ab}\delta g_{ab}}{2}\nabla.\delta\sigma$$

$$= (P_1^T\delta g)_a\delta\sigma^a - \frac{g^{ab}\delta g_{ab}}{2}\nabla.\delta\sigma + \text{ total derivative}$$

and thus, we have (**disagreement with Polchinski**);

$$\int d^2\sigma\sqrt{g}((P_1^T\delta g)_a\delta\sigma^a - g^{ab}\delta g_{ab}\delta\omega) = 0$$

For this to be zero for all diff × Weyl, we need the following;

$$(P_1^T\delta g)_a = 0, \quad g^{ab}\delta g_{ab} = 0$$

which means that δg_{ab} is traceless and the number of solutions of first equation gives us the moduli.
Conformal killing vectors will be acquired if the metric variation due to diff × weyl transformations is zero i.e. (5.2.1) is zero. Contracting it with g^{ab} gives us $\delta\omega$ as follows;

$$\delta\omega = \frac{1}{2}\nabla.\delta\sigma$$

Then, the only equation that we get is $(P_1\delta g)_a = 0$ which is conformal killing equation which gives conformal killing vectors (CKVs). We can vary the metric around the conformal gauge;

$$g_{ab}(z,\bar{z}) = \frac{1}{2}e^{\omega(z,\bar{z})}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

to write the conformal killing equation as follows (**derive this**);

$$\partial\delta g_{\bar{z}\bar{z}} = \bar{\partial}\delta g_{zz} = \partial\delta\bar{z} = \bar{\partial}\delta z = 0$$

Thus, the number of moduli (denoted as μ) is dim(ker P_1^T) and the number of CKVs (denoted as κ) is dim(ker P_1). The Riemann-Roch theorem (which can be seen as an index theorem) states that;

$$\dim(\ker P_1^T) - \dim(\ker P_1) = -3\chi \Rightarrow \mu - \kappa = -3\chi \tag{5.2.2}$$

We will now show that there can't be any CKVs for $\chi < 0$. We first see that by using Weyl transformations, we can bring Ricci curvature to a constant. We have the following relation between old and new R;

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega) \Rightarrow 2\nabla^2\omega = R - e^{2\omega}R'$$

which can be solved for constant R' (**is this a correct argument?**). Moreover, due to Gauss-Bonnet theorem (for no boundaries);

$$\int_M d^2\sigma\sqrt{g}R = 2\pi\chi(M)$$

we see that for a constant R, the sign of R and χ is the same. Now, we derive a result;

$$\begin{aligned}(P_1^T P_1 u)_a &= -\frac{1}{2}\left(\nabla^b\nabla_a u_b + \nabla^b\nabla_b u_a - \nabla^b(g_{ab}\nabla_c u^c)\right) = -\frac{1}{2}\nabla^2 u_a - \frac{1}{2}\left(\nabla^b\nabla_a u_b - \nabla^b(g_{ab}\nabla_c u^c)\right)\\ &= -\frac{1}{2}\nabla^2 u_a + \frac{1}{2}(\nabla_a\nabla_c - \nabla_c\nabla_a)u^c = -\frac{1}{2}\nabla^2 u_a - \frac{1}{4}R u_a\end{aligned} \tag{5.2.3}$$

(Fill in the details of the last step). Using the facts that $(P_1\delta\sigma)_{ab}$ is traceless and symmetric, we can derive the following identity as well;

$$(P_1\delta\sigma)_{ab}(P_1\delta\sigma)^{ab} = \nabla_a\delta\sigma_b\nabla^b\delta\sigma^a = \text{total derivative} - \delta\sigma_b\nabla_a\nabla^b\delta\sigma^a = \text{total derivative} - \delta\sigma_b(P_1^T P_1\delta\sigma)^b$$

where in the last step, I used the fact that $(P_1^T\sigma)^{ab}$ is traceless and symmetric. Now, using (5.2.3), we get the following;

$$\begin{aligned}\int d^2\sigma\sqrt{g}(P_1\delta\sigma)_{ab}(P_1\delta\sigma)^{ab} = \sqrt{g}\int d^2\sigma\delta\sigma_a(P_1^T P_1\delta\sigma)^a &= \int d^2\sigma\sqrt{g}\left(-\delta\sigma^a\frac{1}{2}\nabla^2\delta\sigma_a - \frac{1}{4}R\delta\sigma^a\delta\sigma_a\right)\\ &= \int d^2\sigma\sqrt{g}\left(\frac{1}{2}\nabla_b\delta\sigma^a\nabla^b\delta\sigma_a - \frac{1}{4}R\delta\sigma^a\delta\sigma_a\right)\end{aligned}$$

where in the last term, surface terms are dropped. So, if χ is negative, then R is negative and thus, the integral in the last line is positive definite. Thus, $(P_1\delta\sigma)_{ab}$ can't vanish. Thus, there are no CKVs for negative χ.

5.2.1 Riemann surfaces

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5.3 The measure for moduli

Write details about the derivation. The expression for the S matrix with n momenta $k_1, ..., k_n$ and internal states $j_1, ..., j_n$ is given by as follows;

$$S_{j_1,...j_n}(k_1,...,k_n) = \sum_{\text{comp. top.}}\int_F \frac{d^\mu t}{n_R}\int[d\phi\, db\, dc]e^{-S_m - S_g - \lambda\chi}$$

$$\times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_k \hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)}\, \mathcal{V}_{j_i}(k_i, \sigma_i) \tag{5.3.1}$$

where F is the moduli space, n_R is the order of discrete symmetries, f contains fixed coordinates and the inner product is defined as follows;

$$(b, \partial_k \hat{g}) = \int d^2\sigma \sqrt{g}\, b_{ab} \partial_k \hat{g}^{ab}$$

(Write about the S matrix expression in determinant and RR theorem proof using path integrals)

5.3.1 Expression in terms of determinants

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5.3.2 The Riemann Rock theorem

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5.4 More about the measure

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5.4.1 Gauge invariance

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5.4.2 BRST invariance

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5.4.3 Measure for Riemann surfaces

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6 Chapter 6: Tree level amplitudes

6.1 Riemann surfaces

6.1.1 The sphere

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6.1.2 The disc

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6.1.3 The projective plane

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6.2 Scalar expectation values

We want to derive a differential equation for the green's functions. To do that, we start with the generating functional;

$$Z[J] = \left\langle \exp\left(i \int d^2\sigma J_\mu(\sigma) X^\mu(\sigma) \right) \right\rangle \tag{6.2.1}$$

Now, let's consider the following set of Helmholtz equations;

$$(\partial^2 + \omega_I^2) X_I(\sigma) = 0$$

solutions of which form a complete set (**find more about this**). So, we expand X^μ in terms of X_I as follows;

$$X^\mu = \sum_I x_I^\mu X_I$$

Moreover, since the Helmholtz equation is linear, we can rescale the solutions and the rescaled version would still be a solution and thus, we can choose the following normalization;

$$\int d^2\sqrt{g}\, d^2\sigma X_I X_{I'} = \delta_{II'}$$

The orthogonality of solutions follows from the fact that the Helmholtz equation can be written down as a Strum-Liouville equation. Now, we calculate (6.2.1) as follows;

$$Z[J] = \prod_{\mu,I} \int dx_I^\mu \exp\left(i x_I^\mu \int d^2\sigma J_\mu(\sigma) X_I^\mu(\sigma) \right) \exp\left(-\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \sum_{I'} \omega_{I'}^2 x_I^\mu x_{\mu I'} X_I X_{I'} \right)$$

where I took the I and μ summation out as a product. Using the orthogonality of X_I's, we get the following;

$$Z[J] = \prod_{\mu,I} \int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + i x_I^\mu J_{\mu I} \right)$$

where;

$$J_{\mu I} = \int d^2\sigma J_\mu(\sigma) X_I(\sigma)$$

Now, using the orthonormality of X_I's and using the fact that X_0 is a constant mode, we get;

$$X_0 = \left(\int d^2\sigma \sqrt{g} \right)^{-1/2}$$

Moreover, using the Helmholtz equation, we get $\omega_0^2 = 0$ and thus, all the dx_I^μ integrals in $Z[J]$ are gaussian except the dx_0^μ integrals. Doing these integrals give factors of $2\pi\delta(J_{0\mu})$. So, we do a product over $I \neq 0$ only and doing the μ product gives us $(2\pi)^d \delta^{(d)}(J_0)$. Now, we use the following gaussian integral;

$$\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} \right)$$

we get;

$$\int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + i x_I^\mu J_{\mu I}\right) = \sqrt{\frac{4\pi\alpha'}{w_I^2}} \exp\left(-\pi\alpha' \frac{J_{I\mu} J_{I\mu}}{w_I^2}\right)$$

Doing the μ product, we get the following;

$$\prod_\mu \int dx_I^\mu \exp\left(-\frac{\omega_I^2 x_I^\mu x_{\mu I}}{4\pi\alpha'} + i x_I^\mu J_{\mu I}\right) = \left(\frac{4\pi\alpha'}{w_I^2}\right)^{d/2} \exp\left(-\pi\alpha' \frac{J_I . J_I}{w_I^2}\right)$$

but $\mu = 0$ case will give a problem because this case will give a positive sign in the first term in the exponential in the integrand. So, we can do multiply x_I^0 by i to get the right sign. This will give an additional factor of i in the expression of $Z[J]$ (**look at another reason that is written on page 170**). So, the last expression is as follows;

$$Z[J] = i(2\pi)^d \delta^{(d)}(J_0) \prod_{I\neq 0} \left(\frac{4\pi\alpha'}{w_I^2}\right)^{d/2} \exp\left(-\pi\alpha' \frac{J_I . J_I}{w_I^2}\right)$$

Now, we do the following manipulation;

$$-\pi\alpha' \frac{J_I . J_I}{w_I^2} = -\frac{1}{2}\int d^2\sigma d^2\sigma' \left[J(\sigma)J(\sigma') \times \frac{2\pi\alpha'}{w_I^2} X_I(\sigma) X_I(\sigma')\right]$$

and we use the fact that the eigenvalues of $-\nabla^2$ are ω_I^2 to deduce that the product of nonzero ω_I^2's is $\det(-\nabla^2)$. Using these facts, we get the following;

$$Z[J] = i(2\pi)^d \delta^{(d)}(J_0) \prod_{I\neq 0} \left(\det \frac{-\nabla^2}{4\pi\alpha'}\right)^{d/2} \exp\left(-\frac{1}{2}\int d\sigma\, d\sigma' J(\sigma) J(\sigma') G(\sigma,\sigma')\right)$$

where

$$G(\sigma,\sigma') = \sum_{I\neq 0} \frac{2\pi\alpha'}{\omega_I^2} X_I(\sigma) X_I(\sigma')$$

This Green's function satisfies the following differential equation;

$$\nabla^2 G(\sigma,\sigma') = -2\pi\alpha' \sum_{I\neq 0} X_I(\sigma) X_I(\sigma')$$

To write this equation in a more convenient form, we recall the fact that X_I's form a complete set and thus, we can expand the σ dependence of $\delta(\sigma - \sigma')$ in terms of these functions as follows (we can encode the σ' dependence into the expansion coefficients then);

$$\delta(\sigma - \sigma') = \sum_I x_I(\sigma') X^I(\sigma) \Rightarrow \int \sqrt{g(\sigma)} X^{I'}(\sigma) \delta(\sigma - \sigma') d^2\sigma = \sum_I x_I(\sigma') \int d^2\sigma \sqrt{g(\sigma)} X^I(\sigma) X^{I'}(\sigma)$$

$$\Rightarrow x_{I'}(\sigma') = \sqrt{g(\sigma')} X^{I'}(\sigma') \Rightarrow x_I(\sigma') = \sqrt{g(\sigma')} X^I(\sigma')$$

$$\Rightarrow \delta(\sigma - \sigma') = \sum_I \sqrt{g(\sigma')} X^I(\sigma') X^I(\sigma) \Rightarrow (g(\sigma'))^{-1/2} \delta(\sigma - \sigma') = (X^0)^2 + \sum_{I\neq 0} X^I(\sigma') X^I(\sigma)$$

$$\Rightarrow \sum_{I\neq 0} X^I(\sigma) X^I(\sigma') = (g(\sigma))^{-1/2} \delta(\sigma - \sigma') - (X^0)^2$$

where we used the fact that δ function is even and X_0 is a constant. Therefore, we can deduce that;

$$-\frac{1}{2\pi\alpha'} \nabla^2 G(\sigma,\sigma') = (g(\sigma))^{-1/2} \delta(\sigma - \sigma') - (X^0)^2 \tag{6.2.2}$$

Let's start to work out the expectation values on the sphere. On the sphere, we know that metric and we can derive the solution of (6.2.2) on the sphere **(derive this)**;

$$G(z_1,\bar{z}_1,z_2,\bar{z}_2) = -\frac{\alpha'}{2} \ln |z_{12}|^2 + f(z_1,\bar{z}_1) + f(z_2,\bar{z}_2) \tag{6.2.3}$$

where $f(z,\bar{z})$ is given as follows;

$$f(z,\bar{z}) = \frac{\alpha' X_0^2}{4} \int d^2z' e^{2\omega(z',\bar{z}')} \ln|z-z'| + k$$

where k is a constant that can be determined ((**derive this**). We start with the following expectation value;

$$A^n_{S^2}(k,\sigma) = \langle : \exp(ik_1 X(\sigma_1)) : ... : \exp(ik_n X(\sigma_n)) : \rangle$$

$$= \langle : \exp(ik_1 X(\sigma_1) + ... + ik_n X(\sigma_n)) : \rangle = \left\langle \exp\left[i\left(\sum_j \int d^2\sigma \delta(\sigma - \sigma_j) k_j \right) \right] . X(\sigma_j) \right\rangle$$

which gives us the following expression for $J^\mu(\sigma)$;

$$J^\mu(\sigma) = \sum_j k_j^\mu \delta^{(2)}(\sigma - \sigma_j) \Rightarrow J_I^\mu = \int d^2\sigma \sum_j k_j^\mu \delta^{(2)}(\sigma - \sigma_j) X_I(\sigma) = \sum_j k_j^\mu X_I(\sigma_j)$$

$$\Rightarrow J_0^\mu = X_0 \sum_j k_j^\mu \Rightarrow \delta^{(d)}(J_0) = X_0^{-d} \delta^{(d)}\left(\sum_j k_j^\mu \right)$$

Moreover, we get the following;

$$= -\int d^2\sigma d^2\sigma' J(\sigma).J(\sigma') G(\sigma,\sigma') = -\int d^2\sigma d^2\sigma' \sum_{jl} k_j.k_l G(\sigma,\sigma') \delta(\sigma-\sigma_j)\delta(\sigma-\sigma_l) G(\sigma,\sigma')$$

$$= -\frac{1}{2}\sum_{j,l} k_j.k_l G(\sigma_j,\sigma_l) = -\sum_{j<l} k_j.k_l G(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j) :$$

So, the required amplitude becomes the following;

$$A^n_{S^2}(k,\sigma) = iC^X_{S^2}(2\pi)^d \delta^{(d)}(\sum_j k_j) \exp\left(-\sum_{j<l} k_j.k_l G(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j) : \right)$$

where $C^X_{S^2}$ is given as follows;

$$C^X_{S^2} = X_0^{-d} \left(\det \frac{-\nabla^2}{4\pi\alpha'} \right)^{-d/2}_{S^2}$$

Now, we need to talk about the normal ordered $: G(\sigma,\sigma') :$ operator. It can be shown that (**Use section 3.6 methods**);

$$: G(\sigma,\sigma') := G(\sigma,\sigma') + \frac{\alpha'}{2} \ln d^2(\sigma,\sigma')$$

and by using (6.2.3), we get (**derive this**);

$$: G(\sigma,\sigma) := 2f(\sigma) + \alpha'\omega(\sigma) \Rightarrow : G(z=z', \bar{z}=\bar{z}') := 2f(z,\bar{z}) + \alpha'\omega(z,\bar{z})$$

Now, we do the following manipulation;

$$-\sum_{j<l} k_j.k_l G(\sigma_j,\sigma_l) - \frac{1}{2}\sum_j k_j^2 : G(\sigma_j,\sigma_j) := \frac{\alpha'}{2}\sum_{j<l} k_j.k_l \ln|\sigma_j - \sigma_l|^2 - \sum_{j<l} k_j.k_l f(\sigma_j) - \sum_{l<j} k_j.k_l f(\sigma_l) - \sum_j k_j^2 f(\sigma_j) - \frac{\alpha'}{2}\sum_j k_j^2 \omega(\sigma_j)$$

$$= \frac{\alpha'}{2}\sum_{j<l} k_j.k_l \ln|\sigma_j - \sigma_l|^2 - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j) - \sum_j \left(\sum_{l<j} + \sum_{l>j} + \sum_{l=j} \right) k_l.k_j f(\sigma_j) = \sum_{j<l} \ln|\sigma_j - \sigma_l|^{\alpha' k_j.k_l} - \frac{\alpha'}{2}\sum_j k_j^2 \omega(\sigma_j)$$

$$= \sum_{j<l} \ln|z_{jl}|^{\alpha' k_j.k_l} - \frac{\alpha'}{2}\sum_j k_j^2\omega(\sigma_j)$$

where we interchanged the indices j and l in one of the $k.kf(\sigma)$ terms. So, the amplitude becomes;

$$A^n_{S^2}(k,\sigma) = iC^X_{S^2}(2\pi)^d \delta^{(d)}(\sum_j k_j) \exp\left(-\frac{\alpha'}{2}\sum_j k_j^2 \omega(\sigma_j) \right) \prod_{j<l} |z_{jl}|^{\alpha' k_j k_l} \tag{6.2.4}$$

This result can be generalized to the following;

$$\left\langle \prod_{j=1}^{n} : e^{ik_j X(z_j,\bar{z}_j)} : \prod_{k=1}^{p} \partial X^{\mu_k}(z'_k) \prod_{l=1}^{q} \partial X^{\mu_l}(z''_l) \right\rangle_{S^2} = iC^X_{S^2}(2\pi)^d \delta^{(d)}(\sum_j k_j) \prod_{j<l} |z_{jl}|^{\alpha' k_j k_l}$$

$$\times \left\langle \prod_{j=1}^{p} \partial X^{\mu_j}(z'_j) \prod_{k=1}^{q} \bar{\partial} X^{\mu_j}(\bar{z}''_j) \right\rangle$$

$$= iC^X_{S^2}(2\pi)^d \delta^{(d)}(\sum_j k_j) \prod_{j<l} |z_{jl}|^{\alpha' k_j k_l} \times \left\langle \prod_{j=1}^{p} [v^{\mu_j}(z'_j) + q^{\mu_j}(z'_j)] \prod_{k=1}^{q} [\tilde{v}^{\mu_j}(\bar{z}'_j) + \tilde{q}^{\mu_j}(\bar{z}'_j)] \right\rangle \tag{6.2.5}$$

where $v^{\mu_j}(z'_j)$ and $\tilde{v}^{\mu_j}(z'_j)$ are as follows;

$$v^\mu(z) = -i\frac{\alpha'}{2} \sum_{i=1}^{n} \frac{k_i^\mu}{z - z_i}, \quad \tilde{v}^\mu(\bar{z}) = -i\frac{\alpha'}{2} \sum_{i=1}^{n} \frac{k_i^\mu}{\bar{z} - \bar{z}_i}$$

and $q^\mu(z) = \partial X^\mu(z) - v^\mu(z), \;\; \tilde{q}^\mu(\bar{z}) = \bar{\partial} X^\mu(\bar{z}) - \tilde{v}^\mu(\bar{z})$
(Write more about the holomorphicity methods (already done) and rest of section).

6.2.1 The disc

The green's function for the disc is as follows (**derive this**);

$$G(z_1, \bar{z}_1, z_2, \bar{z}_2) = -\frac{\alpha'}{2} \ln|z_1 - z_2|^2 - -\frac{\alpha'}{2} \ln|z_1 - \bar{z}_2|^2 \tag{6.2.6}$$

where the second term can be seen as the image charge term (**justify that this is required due to the Neumann condition**). The result is as follows (**derive this**);

$$A^n_{D^2}(k,\sigma) = iC^X_{D^2}(2\pi)^d \delta^{(d)}(\sum_j k_j) \prod_{j<l} |z_j - z_l|^{\alpha' k_j k_l} |z_j - \bar{z}_l|^{\alpha' k_j k_l} \prod_i |z_i - z_i|^{\alpha' k_i^2/2} \tag{6.2.7}$$

6.3 The projective plane

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6.4 The bc CFT

6.4.1 The sphere

We evaluate the expectation value of c ghosts on the sphere because of the sphere, there are no moduli and thus, there is no need to evaluate b ghosts and since there are six CKVs on the sphere, the lowest c expectation value that we will need to evaluate is as follows; (**Polchinski says that this is the simplest non-vanishing one. Find justification for this**)

$$\langle c(z_1)c(z_2)c(z_3)\tilde{c}(z_4)\tilde{c}(z_5)\tilde{c}(z_6)\rangle$$

but this $\det C^a_{0j}$ i.e. the c part of the Fadeev-Popov ghost action. Now, C^a_{0j} are CKVs and for the sphere, we have the following C^a_{0j}'s;

$$C^z_{01} = 1, \; C^z_{02} = z, \; C^z_{03} = z^2, \; C^z_{04} = C^z_{05} = C^z_{06} = 0$$

$$, \; C^{\bar{z}}_{01} = C^{\bar{z}}_{02} = C^{\bar{z}}_{03} = 0, \; C^{\bar{z}}_{01} = 1, \; C^{\bar{z}}_{01} = \bar{z}, \; C^{\bar{z}}_{03} = \bar{z}^2$$

but since the basis that we are using here is not an orthonormal basis, we will have a Jacobian and thus, we will have a constant $C^g_{S^2}$ in front of the $(C^{z_i}_{0j}, C^{z_i}_{0j})$ matrix as well (**work this out**). Determinant of the full $(C^{z_i}_{0j}, C^{z_i}_{0j})$ matrix with this constant is as follows;

$$C^g_{S^2} \det \begin{pmatrix} C^{z_1}_{01} & C^{z_2}_{01} & C^{z_3}_{01} & C^{\bar{z}_1}_{01} & C^{\bar{z}_2}_{01} & C^{\bar{z}_3}_{01} \\ C^{z_1}_{02} & C^{z_2}_{02} & C^{z_3}_{02} & C^{\bar{z}_1}_{02} & C^{\bar{z}_2}_{02} & C^{\bar{z}_3}_{02} \\ C^{z_1}_{03} & C^{z_2}_{03} & C^{z_3}_{03} & C^{\bar{z}_1}_{03} & C^{\bar{z}_2}_{03} & C^{\bar{z}_3}_{03} \\ C^{z_1}_{04} & C^{z_2}_{04} & C^{z_3}_{04} & C^{\bar{z}_1}_{04} & C^{\bar{z}_2}_{04} & C^{\bar{z}_3}_{04} \\ C^{z_1}_{05} & C^{z_2}_{05} & C^{z_3}_{05} & C^{\bar{z}_1}_{05} & C^{\bar{z}_2}_{05} & C^{\bar{z}_3}_{05} \\ C^{z_1}_{06} & C^{z_2}_{06} & C^{z_3}_{06} & C^{\bar{z}_1}_{06} & C^{\bar{z}_2}_{06} & C^{\bar{z}_3}_{06} \end{pmatrix} = C^g_{S^2} \det \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 \\ z_1^2 & z_2^2 & z_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \bar{z}_4 & \bar{z}_5 & \bar{z}_6 \\ 0 & 0 & 0 & \bar{z}_4^2 & \bar{z}_5^2 & \bar{z}_6^2 \end{pmatrix}$$

$$= C^g_{S^2} z_{12} z_{23} z_{31} \bar{z}_{45} \bar{z}_{56} \bar{z}_{64} \tag{6.4.1}$$

We can deduce a result with b insertions as well. We should have additional insertions for c ghosts if we have b insertions because we want them to contract with one another. The result is as follows (**derive this**);

$$\left\langle \prod_{i=1}^{p+3} c(z_i) \prod_{j=1}^{p} b(z_i'). \text{ anti} \right\rangle = C^g_{S^2} \frac{z_{p+1,p+2} z_{p+1,p+3} z_{p+2,p+3}}{(z_1 - z_1')...(z_p - z_p')}. \text{ anti} \pm \text{ (permutations)} \tag{6.4.2}$$

We don't care about the overall sign. (**Write the holomorphicity argument and other stuff in the subsection**).

6.4.2 The disc

The doubling trick gives us the following for the disc;

$$\langle c(z_1) c(z_2) c(z_3) \rangle_{D^2} = C^g_{D^2} z_{12} z_{23} z_{31}$$

$$\langle c(z_1) c(z_2) \tilde{c}(\bar{z}_3) \rangle_{D^2} = \langle c(z_1) c(z_2) c(z_3') \rangle_{D^2} = C^g_{D^2} z_{12} (z_2 - \bar{z}_3)(\bar{z}_3 - z_1)$$

6.4.3 The projective plane

Similarly, for the projective plane, we can deduce the following (**derive these**);

$$\langle c(z_1) c(z_2) c(z_3) \rangle_{RP_2} = C^g_{RP_2} z_{12} z_{23} z_{31}$$

$$\langle c(z_1) c(z_2) \tilde{c}(\bar{z}_3) \rangle_{RP_2} = \langle c(z_1) c(z_2) c(z_3') \rangle_{RP_2} = C^g_{RP_2} z_{12} (1 + z_1 \bar{z}_3)(1 + z_2 \bar{z}_3)$$

6.5 The Veneziano amplitude

We calculate the tachyon three function now. The vertex operator for the tachyon is as follows;

$$g_0 \int_{\mathbb{R}} ds \; : e^{ik.X(s)} :$$

where g_0 is the string coupling. We will use the upper half plane for the calculation with y as the coordinate on the boundary. Moreover, we can fix three points on the plane by using three CKVs of the plane. The PSL$(2, \mathbf{R})$ group can't change the cycling and thus, we have to sum over two possible cyclings. There will be no b insertions through (**Find about the absence of metric factors**). The three-point S matrix is thus given as follows;

$$S_{D^2}(k_1, k_2, k_3) = e^{-\lambda} g_0^3 \left\langle {}^*_* c^1 e^{ik_1.X}(y_1) {}^*_*{}^*_* c^1 e^{ik_2.X}(y_2) {}^*_*{}^*_* c^1 e^{ik_3.X}(y_3) {}^*_* \right\rangle_{D_2} + (k_2 \leftrightarrow k_3)$$

where the normal ordering is the boundary normal ordering. Using the boundary expectation value for the disc and (6.4.1), we get;

$$S_{D^2}(k_1, k_2, k_3) = iC_{D_2} (2\pi)^{26} \delta^{26}(\sum_j k_j) g_0^3 |y_{12}|^{1+2\alpha' k_1.k_2} |y_{23}|^{1+2\alpha' k_2.k_3} |y_{31}|^{1+2\alpha' k_1.k_3}$$

$$+ iC_{D_2} (2\pi)^{26} \delta^{26}(\sum_j k_j) g_0^3 |y_{12}|^{1+2\alpha' k_1.k_3} |y_{23}|^{1+2\alpha' k_2.k_3} |y_{31}|^{1+2\alpha' k_1.k_2}$$

where $C_{D_2} = e^{-\lambda C^X_{D_2} C^g_{D_2}}$ Using the massless condition, we have;

$$k_1^2 = k_2^2 = k_3^2 = \frac{1}{\alpha'}$$

because we have tachyons here. Moreover, due to the momentum conservation is as follows;

$$k_1 + k_2 + k_3 = 0$$

Using these two conditions, we derive the following result;

$$(k_1+k_2)^2 = k_3^2 = \frac{1}{\alpha'} \text{ and } (k_1+k_2)^2 = k_1^2+k_2^2+2k_1.k_2 = \frac{2}{\alpha'} + 2k_1.k_2 \Rightarrow 2\alpha' k_1.k_2 + 1 = 0$$

and similar results follow for $2\alpha' k_2.k_3 + 1$ and $2\alpha' k_1.k_3 + 1$. Therefore, the S matrix becomes;

$$S_{D^2}(k_1,k_2,k_3) = 2iC_{D_2} g_0^3 (2\pi)^{26}\delta^{26}(\sum_j k_j) \tag{6.5.1}$$

We calculate the four point tachyon function now. Three points can be fixed and hence, we will have to integrate over the fourth coordinate. We have to sum over all the orderings but we can do that by just interchanging k_1 and k_2. We get the following;

$$S_{D^2}(k_1,k_2,k_3,k_4) = ig_0^4 C_{D^2}(2\pi)^{26}\delta^{(26)}(\sum_j k_j)|y_{12}y_{23}y_{31}| \int dy_4 \prod_{j<l} |y_{jl}|^{2\alpha' k_j.k_l} + (k_2 \leftrightarrow k_3)$$

$$= ig_0^4 C_{D^2}(2\pi)^{26}\delta^{(26)}(\sum_j k_j)|y_{12}y_{23}y_{31}| \int dy_4 \prod_{j<l} |y_{jl}|^{2\alpha' k_j.k_l} + (k_2 \leftrightarrow k_3)$$

where we have the following;

$$2\alpha' k_1.k_2 = -2-2\alpha' s,\ 2\alpha' k_2.k_3 = -2-2\alpha' u,\ 2\alpha' k_1.k_3 = -2-2\alpha' t$$

$$2\alpha' k_2.k_4 = -2-2\alpha' t,\ 2\alpha' k_3.k_4 = -2-2\alpha' s,\ 2\alpha' k_1.k_4 = -2-2\alpha' u$$

where

$$s = (k_1+k_2)^2 = (k_3+k_4)^2,\ t = (k_1+k_3)^2 = (k_2+k_4)^2,\ u = (k_1+k_4)^2 = (k_2+k_3)^2 \Rightarrow s+t+u = -\frac{4}{\alpha'},\ s = -\frac{2}{\alpha'}$$

and thus,the factors dependent on ys that appear in the amplitude are as follows;

$$|y_{12}|^{-1-\alpha' s}|y_{13}|^{-1-\alpha' t}|y_{14}|^{-2-\alpha' u}|y_{23}|^{-1-\alpha' u}|y_{24}|^{-2-\alpha' t}|y_{34}|^{-2-\alpha' s}$$

$$= y_3^{-1-\alpha' t} y_4^{-2-\alpha' u}(1-y_3)^{-1-\alpha' u}(1-y_4)^{-2-\alpha' t}(y_3-y_4)^{-2-\alpha' s}$$

where I set $y_1 = 0, y_1 = 1$. If we now set $y_3 = \infty$, then we see that in the large y_3 limit, the exponent of y_3 is $-4-\alpha'(s+t+u)$ which is zero and thus, we can choose this value of y_3 which effectively put all the y_3 dependent factors to unity. So, the amplitude becomes;

$$ig_0^4 C_{D^2}(2\pi)^{26}\delta^{(26)}(\sum_j k_j)|y_{12}y_{23}y_{31}| \int dy_4 \left[|y_4|^{-2-\alpha' u}(1-y_4)^{-2-\alpha' t} + (t \to s)\right]$$

where the $t \to u$ thing has been introduced it effectively interchanges k_2 and k_3. Now, we split the integral as follows;

$$\int_{-\infty}^0 dy_4 |y_4|^{-2-\alpha' u}(1-y_4)^{-2-\alpha' t} + \int_0^1 dy_4\, y_4^{-2-\alpha' u}(1-y_4)^{-2-\alpha' t} + \int_1^\infty dy_4\, y_4^{-2-\alpha' u}(1-y_4)^{-2-\alpha' t}$$

$$+ \int_{-\infty}^0 dy_4 |y_4|^{-2-\alpha' u}(1-y_4)^{-2-\alpha' s} + \int_0^1 dy_4\, y_4^{-2-\alpha' u}(1-y_4)^{-2-\alpha' s} + \int_1^\infty dy_4\, y_4^{-2-\alpha' u}(1-y_4)^{-2-\alpha' s}$$

We can show using Mobius transformations (**show this**) that the first and fifth integrals give $I(u,s)$, the third and sixth integrals give $I(s,t)$ and the second and fourth integrals give $I(t,u)$ where $I(s,t)$ is given as follows;

$$I(s,t) = \int_0^1 dy\, y^{-2-\alpha' s}(1-y)^{-2-\alpha' t}$$

So, the four-point tachyon function becomes;

$$S_{D_2}(k_1,...k_4) = 2ig_0^4 C_{D^2}(2\pi)^{26}\delta^{(26)}(\sum_j k_j)\left[I(s,t)+I(t,u)+I(u,s)\right] \tag{6.5.2}$$

This integral will converge only if $-2-\alpha' s > -1 \Rightarrow \alpha' s < -1$. At $\alpha' s + 1 = 0$, we can probe the behavior of this integral by looking at the behavior of the leading term as follows;

$$I(s,t) = \int_0^1 dy \left[y^{-2-\alpha' s} + \mathcal{O}(y^{-1-\alpha' s}) \right] = \frac{1}{\alpha' s + 1} + \mathcal{O}(1/(\alpha' s))$$

so we have a simple pole at $s = -1\alpha'$ which is the mass of the tachyon. If we want the pole to have a small imaginary part (for convergence), then we have to use the formula for the principle part of the simple poles as follows;

$$\frac{1}{x - x_0 \mp i\epsilon} = \mathcal{P}\frac{1}{x - x_0} \pm i\pi\delta(x - x_0) \Rightarrow \frac{1}{\alpha' s + 1 \mp i\epsilon} = \mathcal{P}\frac{1}{\alpha' s + 1} \pm i\pi\delta(\alpha' s + 1)$$

Using unitarity, we can relate the four-point function to three-point functions (**derived in chapter 9, fill in the details)**) as follows;

$$S_{D_2}(k_1, ...k_4) = i\int \frac{d^{26}k}{(2\pi)^{26}} \frac{S_{D_2}(k_1, k_2, k) S_{D_2}(-k, k_3, k_4)}{-k^2 + 1/\alpha + i\epsilon} \tag{6.5.3}$$

Using (6.5.1) and (6.5.2), we easily get (by collecting terms singular in $\alpha' s + 1$ from $I(s,t)$ and $I(u,s)$);

$$C_{D_2}\alpha' g_0^2 = 1 \Rightarrow C_{D_2} = \frac{1}{\alpha' g_0^2} \tag{6.5.4}$$

The expressions of determinants on different topologies (using renormalization) agree with the results from unitarity (**enter more details about it**). Using unitarity and the definition of the beta function, we write the four-point function as follows;

$$S_{D_2}(k_1, ...k_4) = \frac{2ig_0^2}{\alpha'}(2\pi)^{26}\delta^{(26)}(\sum_j k_j)\left[B(-\alpha_0(s), -\alpha_0(t)) + B(-\alpha_0(s), -\alpha_0(u))B(-\alpha_0(t), -\alpha_0(u))\right] \tag{6.5.5}$$

where

$$\alpha_0(x) = 1 + \alpha' x,\ B(a,b) = \int_0^1 dy\, y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The Mandelstam variables in terms of center of mass energy E and mass m are easily given as follows (we take k_3^0, k_4^0 negative **(find out more)**);

$$s = E^2,\ t = (4m^2 - E^2)\sin^2\frac{\theta}{2}\ u = (4m^2 - E^2)\cos^2\frac{\theta}{2}$$

where θ is the angle between particle 1 and particle 3. It means that the Regge limit;

$$\text{Regge limit: } s \to \infty,\ t \text{ is fixed}$$

just sends the energy to infinity and the hard scattering limit

$$\text{Hard scattering limit: } s \to \infty,\ t/s \text{ is fixed} \Rightarrow \frac{t}{s} \to -\sin^2\frac{\theta}{2}$$

sends energy to infinity at a fixed angle. Using Stirling approximation, (just using the x^x factor from the approximation of $\Gamma(x+1)$), we see that in the large s limit, we have;

$$B(-\alpha_0(s), -\alpha_0(t)) = \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha_0(t))}{\Gamma(-\alpha' s - \alpha' t - 2)} \sim \frac{s^{-\alpha' s - 1}}{s^{-\alpha' s - \alpha' t - 2}}\Gamma(-\alpha_0(t)) = s^{\alpha' t + 1}\Gamma(-\alpha_0(t)) = s^{\alpha_0(t)}\Gamma(-\alpha_0(t)) \tag{6.5.6}$$

This is the Regge behavior. The hard scattering limit gives (**derive this**);

$$S_{D^2}(k_1, ...k_4) \sim e^{-\alpha' s f(\theta)} \text{ with } f(\theta) = -\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}$$

6.6 Chan-Paton factors and gauge interactions

In 26 dimensions, closed string theory is the unique closed theory (**include argument by Polchinski**). If we want to introduce degrees of freedom on the endpoints of open strings (called Chan Paton degrees of freedom), then a state becomes;

$$|N, k; ij\rangle, \; i, j = 1, ..., N$$

Then, we can define a basis as follows;

$$|N, k, a\rangle = \sum_{ij} |N, k, ij\rangle \lambda^a_{ij}$$

where λ^a_{ij}'s are hermitian matrices with the following normalization;

$$Tr(\lambda^a \lambda^b) = \delta^{ab}$$

These Chan-Paton indices are invariant in Poincare and they don't change the stress tensor. Moreover, they don't change between vertex operator insertions and thus, the adjacent endpoints have to be in the same Chan-Paton state. Thus, traces of Chan Paton factors are introduced in the amplitudes. They are also consistent with the unitarity because of the completeness of $\langle^a$ which implies that (**check this**)

$$\text{Tr}(A\lambda^a)\text{Tr}(B\lambda^a) = \text{Tr}(AB)$$

Using this equation, we can show that (6.5.3) is satisfied.

6.6.1 Gauge interactions

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6.6.2 The unoriented string

The unoriented string requires Ω which does the following

$$\Omega \alpha^\mu_n \Omega^{-1} = (-1)^n \alpha^\mu_n \; \text{(Open strings)}$$

$$\Omega \alpha^\mu_n \Omega^{-1} = \tilde{\alpha}^\mu_n, \; \Omega \tilde{\alpha}^\mu_n \Omega^{-1} = \alpha^\mu_n \; \text{(Closed strings)}$$

Ω does the following to the states;

$$\Omega|N, k\rangle = (-1)^N |N, k\rangle = (-1)^{1+\alpha' m^2} |N, k\rangle = \omega_N |N, k\rangle$$

If we include the Chan Paton indices, we get the following;

$$\Omega|N, k; ij\rangle = \omega_N |N, k; ji\rangle$$

The interchanging Chan Paton indices arise because the worldsheet parity interchanges the string endpoints. Now, we require $\Omega^2 = 1$ (**include the argument that it is conserved in interactions**) and thus, we get the following possibilities;

$\alpha' m^2$	N	λ^a_{ij}	Gauge
Even	Odd	antisymmetric	$SO(N)$
Odd	Even	symmetric	Traceless sym.+singleton

(Write about the traceless representation). The more general spacetime parity is given as follows;

$$\text{Special Case :} \;\; \Omega|N, k; ij\rangle = \omega_N |N, k; ji\rangle = \omega_N \mathbb{I}_{jj'} |N, k; j'i'\rangle \mathbb{I}_{i'i}$$

$$\text{General case :} \;\; \Omega_\gamma |N, k; ij\rangle = \omega_N \gamma_{jj'} |N, k; j'i'\rangle \gamma^{-1}_{i'i}, \;\; \gamma \in U(N)$$

Now, we again want $\Omega^2_\gamma = 1$ (**include the argument**). For that, we compute $\Omega^2_\gamma |N; k; ij\rangle$ as follows;

$$\Omega^2_\gamma |N; k; ij\rangle = \gamma_{jj'} \gamma^{-1}_{i'i} \gamma_{i'i''} \gamma^{-1}_{j''j'} |N; k; i''j''\rangle = ((\gamma^T)^{-1}\gamma)_{ii''} |N; k; i''j''\rangle ((\gamma^T)^{-1}\gamma)^{-1}_{j''j}$$

Now, imposing $\Omega_\gamma^2 = 1$ will give two possibilities (we will justify this choice later);

$$(\gamma^T)^{-1}\gamma = \pm\mathbb{I} \Rightarrow \gamma^T = \pm\gamma$$

Now, to interpret these two possibilities, we get do a $U(N)$ rotation of the Chan Paton factors as follows;

$$|N;k;ij\rangle' = U^{-1}_{ii'}|N;k;i'j'\rangle U_{j'j} \;\; U \in U(N)$$

For the transformed states, we have the following worldhseet parity transformation;

$$\Omega|N;k;ij\rangle' = \gamma'_{jj'}|N;k;j'i'\rangle'(\gamma')^{-1}_{i'i} = (\gamma' U^{-1})_{jj'}|N'k;j'i'\rangle(U\gamma'^{-1})_{i'i}$$

$$\Rightarrow \Omega U^T_{jj'}|N;k;j'i'\rangle(U^T)^{-1}_{i'i} = (\gamma' U^{-1})_{jj'}|N'k;j'i'\rangle(U\gamma'^{-1})_{i'i}$$

$$\Rightarrow \Omega|N;k;ij\rangle = ((U^T)^{-1}\gamma' U^{-1})_{jj'}|N'k;j'i'\rangle(U\gamma'^{-1}U^T)_{i'i} = ((U^T)^{-1}\gamma' U^{-1})_{jj'}|N'k;j'i'\rangle((U^T)^{-1}\gamma' U^{-1})^{-1}_{i'i}$$

So, we see that γ matrices can be written as follows;

$$\gamma = (U^T)^{-1}\gamma' U^{-1} \Rightarrow \gamma' = U^T\gamma U$$

We can use this transformation to set $\gamma' = 1$ (Polchinski says $\gamma = 1$ but this is equivalent) and we see that for $\gamma' = 1$, the equation above talks about the symmetric part of the γ alone. If we call the symmetric and anti symmetric parts as γ_s and γ_a, then we have;

$$U^T\gamma_s U + U^T\gamma_a U = 1 \Rightarrow U^T\gamma_s U - U^T\gamma_a U = 1 \Rightarrow U^T\gamma_s U = 1,\;\; U^T\gamma_a U = 0$$

So, if γ is symmetric, then we can always solve for γ and thus, set $\gamma' = 1$. This gives the trivial worldsheet parity. For γ being anti symmetric, we can set γ' equal to the following (**include more arguments for it**);

$$\gamma' = M = i\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}, \;\; n = 2k$$

(Write the rest of the section)

6.7 Closed string tree amplitudes

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6.7.1 Consistency

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6.7.2 Closed strings on D_2 and RP_2

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6.8 General results

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6.8.1 Mobius invariance

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6.8.2 Path integrals and matrix elements

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6.8.3 Operator calculations

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6.8.4 Relation between inner products

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7 Chapter 7: One loop amplitudes

7.1 Riemann surfaces

The Euler number for genus g surfaces with b boundaries and c crosscaps is as follows;

$$\chi = 2 - 2g - b - c$$

If we want $\chi = 0$, then the following possibilities occur;

g	b	c	Surface	CKVs
1	0	0	Torus	2
0	2	0	Cylinder	1
0	1	1	Mobius Strip	1
0	0	2	Kelin bottle	1

where we also mentioned the number of CKVs (**derive this**).

7.1.1 The torus

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7.1.2 The cylinder

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7.1.3 The Klein bottle

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7.2 CFT on the torus

7.2.1 Scalar correlators

The differential equation in (6.2.2) can be written as follows;

$$\frac{1}{\alpha'}\nabla^2 G(\sigma,\sigma') = -2\pi(g(\sigma))^{-1/2}\delta^{(2)}(\sigma-\sigma') + 2\pi(X_0)^2$$

Now, we know that;

$$X_0^2 = \left(\int d^2\sigma\sqrt{g}\right)^{-1} = \frac{1}{\text{Area}}$$

So, we get the following

$$\frac{1}{\alpha'}\nabla^2 G(\sigma,\sigma') = -2\pi(g(\sigma))^{-1/2}\delta^{(2)}(\sigma-\sigma') + \frac{2\pi}{\text{Area}}$$

We now write this equation in terms of $w, \bar{w}$ coordinates. We see that;

$$w = \sigma^1 + i\sigma^2,\ \bar{w} = \sigma^1 - i\sigma^2 \Rightarrow \partial_1 = \partial + \bar{\partial},\ \partial_2 = i(\partial - \bar{\partial}) \Rightarrow \nabla^2 = \partial_1^2 + \partial_2^2 = 4\partial\bar{\partial}$$

Moreover, we see that;

$$\int d^2\sigma\delta(\sigma-\sigma') = \frac{1}{2}\int d^2w\delta(\sigma-\sigma') = \int d^2w\delta(w-w') \Rightarrow \delta(w-w') = \frac{1}{2}\delta(\sigma-\sigma') \Rightarrow \delta(\sigma-\sigma') = 2\delta(w-w')$$

Lastly, the metric on the torus covering space $(w,\bar{w})$ is flat. Using these results, we get;

$$\frac{2}{\alpha'}\partial\bar{\partial}G(w,\bar{w},w',\bar{w}') = -2\pi\delta^{(2)}(w-w') + \frac{\pi}{\text{Area}}$$

Now, we see that in the covering space, the length of the torus is 2π and height is $2\pi\tau_2$ an thus, the area is $4\pi^2\tau_2$. So, we get;

$$\frac{2}{\alpha'}\partial\bar{\partial}G(w,\bar{w},w',\bar{w}') = -2\pi\delta^{(2)}(w-w') + \frac{1}{4\pi\tau_2}$$

The solution of this equation is (**verify, anti-holomorphic part missing?**);

$$G(w,w',\bar{w},\bar{w}') = -\frac{\alpha'}{2}\ln\left|\vartheta_1\left(\frac{w-w'}{2}\middle|\tau\right)\right|^2 + \frac{\alpha'}{4\pi\tau_2}\left(\text{lm}(w-w')\right)^2 + k(\tau,\tau')$$

$$= -\frac{\alpha'}{2}\ln\left|\vartheta_1\left(\frac{w-w'}{2}\middle|\tau\right)\right|^2 + \frac{\alpha'}{4\pi\tau_2}\left(\text{lm}(w-w')\right)^2 + k(\tau,\tau')$$

This gives us the following expectation value (work out the self contraction factor);

$$\left\langle\prod_{i=1}^{n} : e^{ik_iX(z_i,\bar{z}_i)} :\right\rangle_{T^2} = iC^X_{T^2}(\tau)(2\pi)^d\delta^d\left(\sum_j k_j\right)\times\prod_{i<j}\left|\frac{2\pi}{\partial_\nu\vartheta_1(0|\tau)}\vartheta_1\left(\frac{w_{ij}}{2\pi}\right)\exp\left[-\frac{(\text{Im}w_{ij})^2}{4\pi\tau_2}\right]\right|^{\alpha' k_i.k_j}$$

7.2.2 The scalar partition function

We now compute the partition function as follows;

$$Z(\tau) = \text{Tr}\left[\exp\left(2\pi i\tau_1 P - 2\pi\tau_2 H\right)\right] \text{ where } P = L_0 - \bar{L}_0,\ H = L_0 + \bar{L}_0 - \frac{1}{24}(c+\tilde{c})$$

$$= \text{Tr}(e^{2\pi i(\tau_1+i\tau_2)L_0}e^{-2\pi i(\tau_1-i\tau_2)\tilde{L}_0})e^{2\pi\tau_2\frac{(c+\tilde{c})}{24}}$$

$$= \text{Tr}(q^{L_0}\bar{q}^{\tilde{L}_0})(q\bar{q})^{-\frac{(c+\tilde{c})}{48}} \text{ where } \quad q = e^{2\pi i\tau}, \bar{q} = e^{-2\pi i\bar{\tau}}$$

Now, for bosonic string, the c and $\tilde{c}$ can be the number of dimensions d and thus, we get;

$$Z(\tau) = \text{Tr}(q^{L_0}\bar{q}^{\tilde{L}_0})(q\tilde{q})^{-d/24}$$

To calculate this trace, we need arbitrary states in the theory. They are labelled by the occupation numbers of the modes α^μ_{-n} and $\tilde{\alpha}^\mu_{-n}$ which we will call $N_{\mu n}$ and $\tilde{N}_{\mu n}$. The levels of the state thus then become;

$$N = \sum_{n,\mu} nN_{\mu n},\ \tilde{N} = \sum_{n,\mu} n\tilde{N}_{\mu n}$$

Now, L_0's are given as follows (they won't have the zero mode constant now because these are the modes on the plane **-Is this the right explanation? I guess so-**).

$$L_0 = \frac{\alpha' k^2}{4} + N = \frac{\alpha' k^2}{4} + \sum_{n,\mu} nN_{\mu n} \Rightarrow q^{L_0} = e^{\pi i\tau\alpha' k^2/2}\prod_{\mu n} q^{nN_{\mu n}}$$

$$\tilde{L}_0 = \frac{\alpha' k^2}{4} + \tilde{N} = \frac{\alpha' k^2}{4} + \sum_{n,\mu} n\tilde{N}_{\mu n} \Rightarrow \tilde{q}^{L_0} = e^{-\pi i\tilde{\tau}\alpha' k^2/2}\prod_{\mu n}\bar{q}^{n\tilde{N}_{\mu n}}$$

$$\Rightarrow q^{L_0}\bar{q}^{\tilde{L}_0} = e^{\pi i\tilde{\alpha}' k^2(\tau-\tilde{\tau})/2}\prod_{\mu n} q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}} = e^{-\pi\alpha' k^2\tau_2}\prod_{\mu n} q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}}$$

Please notice that the expressions of q^{L_0} and $\bar{q}^{\bar{L}_0}$ above in terms of the occupation numbers make sense only when they are sandwiched between a state (and the corresponding bra) of a definite set of occupation numbers (i.e. states which are eigenstates of the occupation number operator). Since we are going to do this sandwiching when we take the trace, these expressions are good for us. Now to take the trace, we need to sum over all possible occupancy numbers and integrate over the center of mass momentum. To keep the integration measure dimensionless, we multiply the d^dk measure with the volume of spacetime V_d. So, we get;

$$Z(\tau) = V_d(q\bar{q})^{-d/24}\int\frac{d^dk}{(2\pi)^d}e^{-\pi\alpha' k^2\tau_2}\prod_{\mu,n}\sum_{N_{\mu n},\tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}}$$

Now, we derive the following results;

$$\sum_{N=0}^{\infty} q^{nN} = 1 + q^n + ... = \frac{1}{1-q^n} \Rightarrow \prod_{\mu,n}\sum_{N_{\mu n},\tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}} = \prod_{\mu}\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\frac{1}{1-\bar{q}^n}\right) = \left(\prod_{n=1}^{\infty}\frac{1}{|1-q^n|^2}\right)^d$$

$$= (q\bar{q})^{d/24} \frac{1}{|\eta(\tau)|^{2d}}$$

Moreover,

$$\int \frac{d^d k}{(2\pi)^d} e^{-\pi\alpha' k^2 \tau_2} = \prod_{j=0}^{d-1} \left(\int_{-\infty}^{\infty} \frac{dk^j}{2\pi} e^{-(\pi\alpha'\tau_2) k_j^2} \right)$$

In the above integrals, $j = 0$ integral is a problem because k_0^2 is negative. For this, we do the Wick rotation $k^0 \to ik^d$ and thus, k^0 integral will give a factor of i. So, we get;

$$\int \frac{d^d k}{(2\pi)^d} e^{-\pi\alpha' k^2 \tau_2} = i \prod_{j=1}^{d} \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} = i \left(\frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \right)^d$$

So, the partition function becomes;

$$Z(\tau) = iV_d \left(Z_X(\tau) \right)^d, \quad Z_X(\tau) = \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{|\eta(\tau)|^2}$$

(Include the other arguments for partition function).

7.2.3 The bc CFT

For the bc ghost system with b having the weight of 2, the L_0 mode (as derived in chapter 2), is given as follows;

$$L_0 = -\sum_{n=-\infty}^{\infty} n : b_n c_{-n} : -1 = -\sum_{n=1}^{\infty} n c_{-n} b_n + \sum_{n=1}^{\infty} n b_{-n} c_n - 1 = \sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) - 1$$

Moreover, recall that we have the following anti-commutation relation;

$$\{b_n, c_m\} = \delta_{m+n,0}$$

We can see that $L_0 + 1$ is nothing but the number operator (or the level N operator). We verify it as follows;

$$\sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) b_{-m} |0\rangle = \sum_{n=1}^{\infty} n b_{-n} c_n b_{-m} |0\rangle = m b_{-m} |0\rangle$$

$$\sum_{n=1}^{\infty} n \left(b_{-n} c_n - c_{-n} b_n \right) c_{-m} |0\rangle = -\sum_{n=1}^{\infty} n c_{-n} b_n c_{-m} |0\rangle = m c_{-m} |0\rangle$$

Each term in $L_0 + 1$ tells us about the level generated by a different ghost field. Let's call the occupancy number of b_{-m} and c_{-m} be B_m and C_m (which can only be 0 or 1 due to anti-commutativity)

$$q^{L_0} |\{B_m\}, \{C_n\}\rangle = q^{mB_m + nC_n} q^{-1} |\{B_m\}, \{C_n\}\rangle$$

So, we see that sandwiching between states of definite B_m and C_n quantum numbers will give us factors like

$$q^{mB_m + nC_n - 1} \text{ and } \bar{q}^{m\bar{B}_m + n\bar{C}_n - 1}$$

Now, summing over all states, we get;

$$\mathrm{Tr}(q^{L_0} \bar{q}^{\bar{L}_0}) = (q\bar{q})^{-1} \prod_{m=1}^{\infty} \sum_{B_m, \tilde{B}_m = 0}^{1} q^{mB_m} \bar{q}^{m\tilde{B}_m} \prod_{m=0}^{\infty} \sum_{C_m, \tilde{C}_m = 0}^{1} q^{mC_m} \bar{q}^{m\tilde{C}_m}$$

$$= (q\bar{q})^{-1} \prod_{m=1} (1 + q^m)(1 + \bar{q}^m) \prod_{m=0} (1 + q^m)(1 + \bar{q}^m) = 4 \prod_{m=1}^{\infty} |1 + q^m|^4$$

For c, the product was started from $m = 0$ because c_0 is a creaation operator but b_0 isn't. For bc ghost system, we have $c = \tilde{c} = -26$ and thus, the partition function becomes;

$$Z(\tau) = 4(q\bar{q})^{13/12} (q\bar{q})^{-1} \prod_{m=1}^{\infty} |1 + q^m|^4 = 4(q\bar{q})^{1/12} \prod_{m=1}^{\infty} |1 + q^m|^4$$

We see that if we had calculated

$$(q\bar{q})^{-(c+\tilde{c})/48}\,\mathrm{Tr}\left[(-1)^F q^{L_0}\bar{q}^{\tilde{L}_0}\right]$$

then due to the c_0 zero modes, we will get a factor of

$$(1-q^0)(1-\bar{q}^0)=0$$

and thus, this quantity vanishes. (**Include the reasoning that this second partition function is the actual quantity that calculates the partition function for the ghosts. Include the anti-commutativity argument.**) Since the torus has one metric modulus and one complex CKV, we need to insert one b and one c in the expectation values on the torus. So, the simplest ghost correlation function that we will need is as follows;

$$\langle c(w_1)b(w_2)\tilde{c}(w_3)\tilde{b}(w_4)\rangle$$

(**Show that only the zero modes contribute**). The zero mode piece will project out the states that have occupancy numbers $C_0=\tilde{C}_0=1$ which is also the state $|\uparrow\uparrow\rangle$. So, we get;

$$\mathrm{Tr}((-1)^F c_0b_0\tilde{c}_0\tilde{b}_0q^{L_0}\bar{q}^{\bar{L}_0})=(q\bar{q})^{1/12}\prod_{m=1}^{\infty}(1-q^m)(1-\bar{q}^m)\prod_{m=1}^{\infty}(1-q^m)(1-\bar{q}^m)=(q\bar{q})^{1/12}\prod_{m=1}^{\infty}|1-q^m|^4$$

7.2.4 General CFTs

Only one simple result is derived. For a general CFT, the eigenvalue of L_0 is the conformal weight. So, the general partition function is as follows;

$$Z(\tau)=\sum_i q^{h_i-c/24}\bar{q}^{\bar{h}_i-\bar{c}/24}(-1)^{F_i}$$

Now, let $\tau=il$ and then, we have;

$$Z(il)=\sum_i e^{-2\pi l\left(h_i+\bar{h}_i-(c+\bar{c})/24\right)}(-1)^{F_i}$$

Now, we have;

$$\tau\to\tau+1\Rightarrow il\to il+1\Rightarrow l\to l-i$$

and thus, this transformation gives the following phase in the partition function;

$$\exp\left[2\pi i\left(h_i+\bar{h}_i-(c+\bar{c})/24\right)\right]$$

This shows that to retain modular invariance, we should have;

$$h_i+\bar{h}_i-\frac{c+\bar{c}}{24}\in\mathbb{Z}\forall i$$

However, for the unity operator, we have $h_i=\bar{h}_i=0$ and thus, $c+\bar{c}$ should be a multiple of 24.
For the argument that follows, we assume that all the conformal weights are non-negative (i.e. the CFT is unitary). Now, if $l\to 0$, then the exponential in the partition function becomes unity and the partition function is determined by the density of states. Now, usually density of states increases with increasing weight (**this is an underlying assumption for this argument**) and thus, the partition function is dominated by the density of states for high h_i. Now, we also have the following constraint;

$$Z(il)=Z(\tau)=Z\left(-\frac{1}{\tau}\right)=Z\left(\frac{i}{l}\right)$$

Thus, we have;

$$Z(il)=Z\left(\frac{i}{l}\right)=\sum_i\exp\left[-\frac{2\pi}{l}\left(h_i+\bar{h}_i-(c+\bar{c})/24\right)\right](-1)^{F_i}$$

Now, if $l\to 0$, then the exponential in $Z(i/l)$ is dominated by lowest $h_i+\bar{h}_i$ which is 0 for the unity operator. So, we have;

$$\lim_{l\to 0}Z\left(\frac{i}{l}\right)\sim\exp\left[\frac{\pi(c+\bar{c})}{12l}\right]$$

So, we can say that;

$$\lim_{l\to 0}Z(il)\sim\exp\left[\frac{\pi(c+\bar{c})}{12l}\right]$$

7.2.5 Theta Functions

The definition of theta function is as follows;

$$\vartheta(\nu,\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau+2\pi i n\nu)$$

We can readily derive the following;

$$\vartheta(\nu+1,\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau+2\pi i n\nu+2\pi i)=\sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau+2\pi i n\nu)=\vartheta(\nu,\tau)$$

Moreover, we can derive the following;

$$\vartheta(\nu+\tau,\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau+2\pi i n\nu+2\pi i n\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau+2\pi i n\nu+2\pi i n\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi i(n^2+2n)\tau+2\pi i n\nu)$$

$$=\exp(-\pi i\tau-2\pi i\nu)\sum_{m=-\infty}^{\infty}\exp(\pi i m^2\tau+2\pi i m\nu)=\exp(-\pi i\tau-2\pi i\nu)\vartheta(\nu,\tau)\quad\text{where } m=n+1$$

The transformation under modular transformations is as follows;

$$\vartheta(\nu,\tau+1)=\sum_{n=-\infty}^{\infty}\exp\left(\pi i n^2\tau+\pi i n^2+2\pi i n\nu\right)$$

(**complete this proof**). The other modular transformation is as follows;

$$\vartheta\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right)=\sum_n\exp\left(-\frac{\pi i n^2}{\tau}+\frac{2\pi i n\nu}{\tau}\right)=\sqrt{-i\tau}\exp\left[i\pi\tau\left(k+\frac{\nu}{\tau}\right)^2\right]$$

$$=\sqrt{-i\tau}\exp\left(\frac{i\pi\nu^2}{\tau}\right)\sum_k\exp\left[i\pi\tau k^2+2\pi i\nu k\right]=\sqrt{-i\tau}\exp\left(\frac{i\pi\nu^2}{\tau}\right)\vartheta(\nu,\tau)$$

where we used the Poisson resummation formula. The zero of ϑ is as follows;

$$\vartheta\left(\nu=\frac{\tau+1}{2},\tau\right)=\sum_n\exp\left(\pi i\tau(n^2+n)+\pi i n\right)=\sum_{n\text{ even}}\exp\left(\pi i\tau(n^2+n)\right)-\sum_{n\text{ odd}}\exp\left(\pi i\tau(n^2+n)\right)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left(\pi i\tau(4n^2+2n)\right)-\sum_{n=-\infty}^{\infty}\exp\left(\pi i\tau((2n-1)^2+2n-1)\right)$$

$$=\sum_{n=-\infty}^{\infty}\exp\left(2\pi i\tau(2n^2+n)\right)-\sum_{n=-\infty}^{\infty}\exp\left(2\pi i\tau(2n^2-n)\right)=2i\sum_{n=-\infty}^{\infty}\exp\left(4\pi i\tau n^2\right)\sin\left(2\pi i\tau n\right)=0$$

The infinite product representation is easily derived using Jacobi triple product identity;

$$\prod_{m=1}^{\infty}(1-q^m)(1+q^{m-1/2}z)(1+q^{m-1/2}z^{-1})=\sum_{n\in\mathbb{Z}}q^{n^2/2}z^n \tag{7.2.1}$$

Setting $q=e^{2\pi i\tau}$ and $z=e^{2\pi i\nu}$, the right hand side of (7.2.1) becomes $\vartheta(\nu,\tau)$. Using the characteristics definition of theta functions;

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau)=\sum_{n=-\infty}^{\infty}\exp\left(\pi i(n+a)^2\tau+2\pi i(n+a)(\nu+b)\right)$$

We obtain, the following (with more notations included);

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\nu,\tau)=\vartheta_{00}(\nu,\tau)=\vartheta_3(\nu|\tau)=\vartheta(\nu,\tau)=\sum_{n\in\mathbb{Z}}q^{n^2/2}z^n$$

$$\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\nu,\tau)=\vartheta_{01}(\nu,\tau)=\vartheta_4(\nu|\tau)=\sum_{n\in\mathbb{Z}}\exp\left(\pi i n^2\tau+2\pi i n\left(\nu+\frac{1}{2}\right)\right)=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}z^n$$

$$\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\nu,\tau) = \vartheta_{10}(\nu,\tau) = \vartheta_2(\nu|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\nu\right)$$
$$= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} z^{n+\frac{1}{2}} = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} z^{n-\frac{1}{2}}$$

where in the last step, we shifted the index from n to $n-1$. Lastly, we have;

$$\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau) = \vartheta_{11}(\nu,\tau) = -\vartheta_1(\nu|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(\nu+\frac{1}{2}\right)\right)$$
$$= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)} z^{n+\frac{1}{2}} e^{\pi\left(n+\frac{1}{2}\right)} = i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)} z^{n+\frac{1}{2}} = -i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)} z^{n-\frac{1}{2}}$$

where we again shifted the index. Now, we derive the modular transformations of these theta functions as follows;

$$\vartheta_3(\nu|\tau+1) = \sum_{n\in\mathbb{Z}} q^{n^2/2} e^{\pi i n^2} z^n = \sum_{n\in\mathbb{Z}} q^{n^2/2} e^{\pi i n} z^n = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} z^n = \vartheta_4(\nu|\tau)$$

where we used the fact that n^2 is odd iff n is odd and n^2 is even iff n is even and thus, we have $e^{\pi i n^2} = e^{\pi i n}$. Continuing, we have;

$$\vartheta_4(\nu|\tau+1) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} e^{\pi i n^2} z^n = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} e^{\pi i n} z^n = \sum_{n\in\mathbb{Z}} q^{n^2/2} z^n = \vartheta_3(\nu|\tau)$$

$$\vartheta_2(\nu|\tau+1) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{\pi i\left(n-\frac{1}{2}\right)^2} z^{n-1/2} = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{\pi i(n^2-n)+\pi i/4} z^{n-1/2}$$
$$= e^{\pi i/4}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} z^{n-1/2} = e^{\pi i/4}\vartheta_2(\nu|\tau)$$

$$\vartheta_1(\nu|\tau+1) = i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{\pi i\left(n-\frac{1}{2}\right)^2} z^{n-1/2} = i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{\pi i(n^2-n)+\pi i/4} z^{n-1/2}$$
$$= ie^{\pi i/4}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} z^{n-1/2} = e^{\pi i/4}\vartheta_1(\nu|\tau)$$

The S transformations are done as follows;

$$\vartheta_3\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right) = \sum_{n\in\mathbb{Z}} e^{-\pi i n^2/\tau + 2\pi i n\nu/\tau} = \sqrt{-i\tau}\sum_{n\in\mathbb{Z}}\exp\left[-i\pi\tau\left(k+\frac{\nu}{\tau}\right)^2\right] = \sqrt{-i\tau}e^{i\pi\nu^2/\tau}\sum_{k\in\mathbb{Z}} q^{k^2/2}z^k = \sqrt{-i\tau}e^{i\pi\nu^2/\tau}\vartheta_3(\nu|\tau)$$

where we used the Poisson resummation formula. The others can be done similarly (**do the others**). The Jacobi identity of the theta functions is as follows;

$$\vartheta_3^4(0|\tau) = \vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau) \tag{7.2.2}$$

(**Prove this**). Prom the product representation, it can easily be seen that $\vartheta_1(0|\tau) = 0$ as follows;

$$\vartheta_1(\nu|\tau) \propto \sin(\pi\nu) \Rightarrow \vartheta_1(0|\tau) = 0 \tag{7.2.3}$$

We again used the fact that $n^2 - n$ is always even (i.e. for all $n \in \mathbb{Z}$). The modular transformation of the eta function is calculated now. We have the following;

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \Rightarrow \eta(\tau+1) = q^{1/24}e^{\pi i/12}\prod_{n=1}^{\infty}(1-q^n e^{2\pi i n}) = e^{\pi i/12}\eta(\tau)$$

To derive the S transformation, we first do another calculation. We will need the product representations of the theta functions that can be derived using (7.2.1) (**derive them**). We proceed as follows;

$$\frac{\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{\eta^3(\tau)} = \frac{2e^{\pi i\tau/4}\prod_{n=1}^{\infty}(1-q^m)^3(1+q^m)^2\left(1-q^{m-\frac{1}{2}}\right)^2\left(1+q^{m-\frac{1}{2}}\right)^2}{e^{\pi i\tau/4}\prod_{n=1}^{\infty}(1-q^n)^3}$$

$$\Rightarrow \sqrt{\frac{\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{2\eta^3(\tau)}} = \prod_{n=1}^{\infty}(1+q^m)\left(1-q^{2m-1}\right)$$

Now, we derive an identity to proceed;

$$\left(1-q^{2m-1}\right)\left(1-q^{2m}\right) = 1-q^{2m}-q^{2m-1}+q^{4m-1} = 1-\left(e^{2\pi i(2m)}\right)^{\tau}-\left(e^{2(2m-1)\pi i}\right)^{\tau}+\left(e^{2(4m-1)\pi i}\right)^{\tau}$$

$$= 1-(1)^{\tau}-(-1)^{\tau}+(-1)^{\tau} = 1-(1)^{\tau} = 1-\left(e^{2\pi i m}\right)^{\tau} = 1-q^m \Rightarrow 1-q^{2m-1} = \frac{1-q^m}{1-q^{2m}}$$

So, we get the following;

$$\sqrt{\frac{\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{2\eta^3(\tau)}} = \prod_{n=1}^{\infty}(1+q^m)\frac{1-q^m}{1-q^{2m}} = \prod_{n=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m}} = 1$$

Thus, we readily have the following;

$$\eta(\tau) = \sqrt[3]{\frac{\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{2}}$$

$$\Rightarrow \eta\left(-\frac{1}{\tau}\right) = \sqrt[3]{\frac{(-i\tau)^{3/2}\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{2}} = \sqrt{-i\tau}\sqrt[3]{\frac{\vartheta_3(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|\tau)}{2}} = \sqrt{-i\tau}\ \ \eta(\tau)$$

This implies the following;

$$\bar{\eta}\left(-\frac{1}{\bar{\tau}}\right) = \sqrt{i\bar{\tau}}\ \ \bar{\eta}(\bar{\tau}) \Rightarrow |\eta(\tau)|^2 = \frac{1}{|\tau|}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2$$

We also note that;

$$\tau_1+i\tau_2 = \tau \to -\frac{1}{\tau} = -\frac{1}{\tau_1+i\tau_2} = -\frac{\tau_1}{|\tau|^2}+i\frac{\tau_2}{|\tau|^2} \Rightarrow \tau_2 \to \frac{\tau_2}{|\tau|^2} \Rightarrow \sqrt{\tau_2} \to \frac{\sqrt{\tau_2}}{|\tau|} \Rightarrow \sqrt{\tau_2} = |\tau|\sqrt{\tau_2'} \tag{7.2.4}$$

where τ_2' is the transformed τ_2. This means that $\sqrt{\tau_2}|\eta(\tau)|^2$ is a modular invariant quantity (note that in T transformation, τ_2 is invariant).

7.3 The torus amplitude

Since the torus metric is invariant under the transformation

$$\sigma^a \to -\sigma^a \Rightarrow w \to -w,\ \bar{w} \to -\bar{w}$$

The torus has a symmetry group of $\mathbb{Z}_2$ and thus, we will have a factor of 2 in the denominator of the moduli integration. Only one CKV is there for the torus and therefore, we insert one $c\tilde{c}$ insertion. Because of one complex modulus, we have one $B\tilde{B}$ insertion where B is given as follows;

$$B = \frac{1}{4\pi}(b,\partial_\tau g) = \frac{1}{4\pi}\int d^2w b_{ww}\partial_\tau g_{ww} = 2\pi i b_{ww}(0)$$

(**Include the metric derivative and the argument for $b_w w$ being evaluated at zero**). So, the torus amplitude becomes;

$$S(1,...,n)_{T^2} = \frac{1}{2}\int d\tau d\bar{\tau}\left\langle B\tilde{B}c\tilde{c}\mathcal{V}(w_1,\bar{w}_1)\prod_{i=2}^{n}\int dw_i d\bar{w}_i\mathcal{V}_i(w_i,\bar{w}_i)\right\rangle_{T^2}$$

$$= 2\pi^2\int d\tau d\bar{\tau}\left\langle b(0)\tilde{b}(0)c\tilde{c}\mathcal{V}(w_1,\bar{w}_1)\prod_{i=2}^{n}\int dw_i d\bar{w}_i\mathcal{V}_i(w_i,\bar{w}_i)\right\rangle_{T^2}$$

We can average over the position w_1 (**include the argument that again, c(0) will pop up**). It goes as follows;

$$\int\frac{d\sigma_1 d\sigma_2}{\text{Area of torus}}c(w_1)\tilde{c}(\bar{w}_1)\mathcal{V}(w_1,\bar{w}_1) = \int\frac{dw_1 d\bar{w}_1}{2\text{Area of torus}}c(w_1)\tilde{c}(\bar{w}_1)\mathcal{V}(w_1,\bar{w}_1)$$

$$= c(0)\tilde{c}(0)\int \frac{dw_1 d\bar{w}_1}{2(2\pi)^2\tau_2}\mathcal{V}(w_1,\bar{w}_1)$$

So, torus amplitude becomes;

$$S_{T^2}(1,...,n) = \int \frac{d\tau d\bar{\tau}}{4\tau_2}\left\langle b(0)\tilde{b}(0)c(0)\tilde{c}(0)\prod_{i=1}^{n}\int dw_i d\bar{w}_i \mathcal{V}_i(w_i,\bar{w}_i)\right\rangle_{T^2}$$

We first calculate the amplitude without any vertex operators and include the matter partition function for 26 dimensions, we get;

$$Z_{T^2} = \int \frac{d\tau d\bar{\tau}}{4\tau_2}\left\langle b(0)\tilde{b}(0)c(0)\tilde{c}(0)\right\rangle_{T^2} iV_{26}(Z_X)^{26} = iV_{26}\int \frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi\alpha'\tau_2)^{-13}|\eta(\tau)|^{4-2\times 26}$$

$$= iV_{26}\int \frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48}$$

For a general CFT, we can write down the partition function similarly by integrating over the COM momentum (with d non-compact dimensions and where $d \geq 2$ so that ghosts can remove the oscillators corresponding to these directions) as follows;

$$Z_{T^2} = V_d\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}\int \frac{d^dk}{(2\pi)^d}e^{-\pi\tau_2\alpha' k^2}\sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{h_i-1} = iV_d\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi\alpha'\tau_2)^{-d/2}\sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{h_i-1} \tag{7.3.1}$$

The steps are like the torus partition function again. The momenta appear in the integrand because $\mathcal{H}_\perp$ doesn't include non-compact momenta, ghosts, and $\mu = 0,1$ (where $\mu = 0,1$ are taken to the non-compact directions). The field theory result for particles paths with circle topology is (**derive this**);

$$Z_{S_1}(m^2) = V_d\int \frac{d^dk}{(2\pi)^d}\int_0^\infty \frac{dl}{2l}e^{-(k^2+m^2)l/2} = V_d\int \frac{d^dk}{(2\pi)^d}\int_0^\infty \frac{dl}{2l}e^{-k^2l/2}e^{-m^2l/2}$$

$$= iV_d\int_0^\infty \frac{dl}{2l}\left(\int \frac{dk_j}{2\pi}e^{-k_j^2l/2}\right)^d e^{-m^2l/2} = iV_d\int_0^\infty \frac{dl}{2l}(2\pi l)^{-d/2}e^{-m^2l/2}$$

where again, the factor of i appears because we need to do wick rotation $k^0 \to ik^0$ for the k^0 integral to converge (k_j's are just integration variables but k^μ are actual non-compact momenta). Now, we sum this partition function over the string mass spectrum which is as follows;

$$m^2 = \frac{2}{\alpha'}(h+\bar{h}-2) \Rightarrow \frac{m^2l}{2} = \frac{l}{\alpha'}(h+\bar{h}-2)$$

Notice that the formula this time has h instead of the level N (**this seems fine but find more justification for this**). However, we also have the level matching condition i.e. $h = \tilde{h}$. We can impose this condition as follows;

$$\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{i(h-\bar{h})\theta} = \delta(h-\bar{h}) = \delta_{h,\bar{h}}$$

The last step makes sense only if $h-\bar{h}$ is an integer but we saw before that due to modular invariance, this is the case and thus, there is no problem in the last step. So, we get;

$$\sum_{i\in\mathcal{H}_\perp} Z_{S^1}(m_i^2) = iV_d\int_0^\infty \frac{dl}{2l}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}(2\pi l)^{-d/2}\sum_{i\in\mathcal{H}_\perp} e^{-l(h_i+\bar{h}_i-2)/\alpha'+i(h_i-\bar{h}_i)\theta}$$

Now we change the variables. The new variables are as follows;

$$\tau = \frac{\theta}{2\pi}+\frac{il}{2\pi\alpha'},\ \bar{\tau} = \frac{\theta}{2\pi}-\frac{il}{2\pi\alpha'} \Rightarrow \theta = \pi(\tau+\bar{\tau}),\ \ l = -i(\tau-\bar{\tau})\pi\alpha' \tag{7.3.2}$$

Let's calculate the range of these new coordinates. The real and imaginary parts of τ are in the following ranges;

$$R:\ -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2},\ \ 0 \leq \tau_2 < \infty$$

We have called this region R. The Jacobian is calculated as follows;

$$d\theta dl = \begin{vmatrix} \partial\theta/\partial\tau & \partial\theta/\partial\bar{\tau} \\ \partial l/\partial\tau & \partial l/\partial\bar{\tau} \end{vmatrix} d\tau d\bar{\tau} = 2\pi^2\alpha' d\tau d\bar{\tau}$$

Moreover, we can also see that $l = 2\pi\alpha'\tau_2$. Lastly, we see that;

$$-l(h_i + \bar{h}_i - 2)/\alpha' + i(h_i - \bar{h}_i)\theta = \frac{i(\tau - \bar{\tau})\pi\alpha'}{\alpha'}(h_i + \bar{h}_i - 2) + i\pi(\tau + \bar{\tau})(h_i - \bar{h}_i) = 2i\pi(h_i - 1)\tau - 2\pi i(\bar{h}_i - 1)\bar{\tau}$$

$$\Rightarrow e^{-l(h_i+\bar{h}_i-2)/\alpha' + i(h_i-\bar{h}_i)\theta} = e^{2i\pi(h_i-1)\tau - 2\pi i(\bar{h}_i-1)\bar{\tau}} = q^{h_i-1}\bar{q}^{\bar{h}_i-1}$$

So, in these new variables, we get;

$$\sum_{i\in\mathcal{H}_\perp} Z_{S^1}(m_i^2) = iV_d \int_R \frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-d/2} \sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{\bar{h}_i-1} \tag{7.3.3}$$

The only difference between this partition function and the actual string partition function is in the domain of τ integration. Now, let's study divergences. Firstly, we see that;

$$Z_{S_1}(m^2) \sim \int_0^\infty \frac{dl}{l^{d/2+1}} = -\frac{2}{d}\left|\frac{1}{l^{d/2}}\right|_0^\infty = \infty \quad (d \geq 2)$$

Notice that the divergence comes due to the fact that $l = 0$ is in the domain. Summing over string spectra gives a sum of divergent terms (as all of the terms in (7.3.3) have the same sign) and thus, it is also divergent. Notice from (7.3.2) that $l = 0$ means that $\tau = \bar{\tau}$ i.e. τ is on the real axis. This axis is absent from F_0 and thus, the UV divergent region is absent.
Let's investigate the $\tau_2 \to \infty$ region for (7.3.1) now. For that, let's change the variables as follows;

$$\tau = \tau_1 + i\tau_2, \quad \bar{\tau} = \tau_1 - i\tau_2 \Rightarrow d\tau d\bar{\tau} = 2d\tau_1 d\tau_2$$

We also do the following manipulation;

$$q^{h_i-1}\bar{q}^{\bar{h}_i-1} = e^{2\pi i(h_i-\bar{h}_i)\tau_1} e^{-2\pi(h_i+\bar{h}_i-2)\tau_2}$$

Now, using these variables, we can write (7.3.1) for $d = 26$ as follows;

$$iV_{26}\int \frac{d\tau_1 d\tau_2}{2\tau_2}(4\pi\alpha'\tau_2)^{-13} \sum_{i\in\mathcal{H}_\perp} e^{2\pi i(h_i-\bar{h}_i)\tau_1} e^{-2\pi(h_i+\bar{h}_i-2)\tau_2} = iV_{26}\int_{\tau_{2\text{lower}}}^\infty \frac{d\tau_2}{2\tau_2}(4\pi\alpha'\tau_2)^{-13} \sum_{i\in\mathcal{H}_\perp} e^{-4\pi(h_i-1)\tau_2}$$

where the τ_1 integration just ensures that $h_i = \bar{h}_i$. Now, we can sum over different h_i (which is just level i.e. N). For $h_i = 0$, we have only one state i.e. the tachyon. For $h_i = 1$, we have 24^2 states because the h_i state contains the following creation operators on the vacuum;

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j, \quad i, j \in \{1, ..., 24\}$$

So, the expansion of (7.3.1) is as follows;

$$iV_{26}\int_{\tau_{2\text{lower}}}^\infty \frac{d\tau_2}{2\tau_2}(4\pi\alpha'\tau_2)^{-13}\left(e^{4\pi\tau_2} + (24)^2 + ...\right)$$

The first tachyon term diverges. Using the fact that;

$$m_i^2 = \frac{2}{\alpha'}(h_i + \bar{h}_i - 2) = \frac{4}{\alpha'}(h_i - 1) \Rightarrow 4(h_i - 1) = \alpha' m_i^2$$

(where the level matching condition is used i.e. $h_i = \bar{h}_i$) we can write down the partition function in the form above for $d \neq 26$ as well. It is as follows;

$$iV_d\int_{\tau_{2\text{lower}}}^\infty \frac{d\tau_2}{2\tau_2}(4\pi\alpha'\tau_2)^{-d/2} \sum_{i\in\mathcal{H}_\perp} \exp\left(-\pi\alpha' m_i^2 \tau_2\right)$$

7.3.1 Physics of the vacuum amplitude

Since $Z_{S_1}(m^2)$ is the vacuum partition function for a circle topology path, the total vaccum amplitude will comprise of all possible number of such paths (but don't count a configuration more than once). SO, we get;

$$Z_{\text{vac}}(m^2) = 1 + Z_{S^1}(m^2) + \frac{1}{2!}\left(Z_{S^1}(m^2)\right)^2 + ... = \exp\left[Z_{S^1}(m^2)\right]$$

where $1/n!$ is added to avoid over counting the contributions. The vaccum amplitude also has another expression which is as follows;

$$Z_{\text{vac}}(m^2) = \langle 0|e^{-iHT}|0\rangle = \langle 0|e^{-i\rho_0 VT}|0\rangle = \langle 0|e^{-i\rho_0 V_d}|0\rangle = e^{-i\rho_0 V_d}$$

where ρ_0 is the vaccum energy density (which should be a constant to avoid breaking Lorentz invariance), V is the volume of the space, T is time and V_d is the volume of spacetime. Now, comparing the above two expressions for the vaccum amplitude, we get;

$$Z_{S^1}(m^2) = -i\rho_0 V_d \Rightarrow \rho_0 = \frac{i}{V_d} Z_{S_1}(m^2)$$

(**Include the cut-off discussion**). The generalization to particles with different spins is as follows (**well motivated but find more justification**);

$$\rho_0 = \frac{i}{V_d}\sum_j (-1)^{\mathbf{F}_j} Z_{S_1}(m_j^2) \tag{7.3.4}$$

7.4 Open and unoriented one loop graphs

7.4.1 The cylinder

The cylinder partition function is relevant for the open strings. The cylinder has one modulus that we call t. The height of the cylinder is $2\pi t$ and the circumference is π. So, we have to calculate;

$$Z_{C_2} = \frac{1}{2}.\frac{1}{2}(2\pi)^2 \int_0^\infty \frac{dt}{2\pi^2 t} \operatorname{Tr}_0'\left[e^{-2\pi t L_0}\right] = \int_0^\infty \frac{dt}{2t} \operatorname{Tr}_0'\left[e^{-2\pi t L_0}\right]$$

where the first half comes due to the Jacobian of $d\sigma^1 d\sigma^2 \to dw d\bar{w}$ transformation, the second one due to the $\mathbb{Z}_2$ symmetry of the cylinder, $(2\pi)^2$ from the B insertions (although they aren't here, they were used in torus case to have $b(0)$ in the expectation value instead of B) and the $2\pi^2 t$ comes from the area of the cylinder. L_0 is the open string Hamiltonian and it is given as follows;

$$L_0 = \alpha' p^2 + N$$

Now, we sum over all possible states and integrate over the momentum to get;

$$Z_{C_2} = V_d n^2 \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t\alpha' k^2} \int_0^\infty \frac{dt}{2t} \prod_{\mu,n}\sum_{N_{\mu n}} e^{-2\pi t n N_{\mu n}}$$

$$= iV_d n^2 \left(8\pi^2\alpha' t\right)^{-d/2} \int_0^\infty \frac{dt}{2t} \prod_{\mu,n}\sum_{N_{\mu n},\mathcal{H}^\perp} e^{-2\pi t n N_{\mu n}} = in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \left[\prod_{n=1}^\infty \frac{1}{1-e^{-2\pi t n}}\right]$$

$$= in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \left[\prod_{n=1}^\infty \frac{1}{1-e^{2i\pi(it)n}}\right] = in^2 \int_0^\infty \frac{dt}{2t} V_d \left(8\pi^2\alpha' t\right)^{-d/2} \eta(it)^{-(d-2)} \tag{7.4.1}$$

where in the second line $\mathcal{H}^\perp$ means that we sum over transverse excitations. The n^2 factor comes due to the Chan Paton degrees of freedom. The $d-2$ comes because ghosts remove oscillators corresponding to two non-compact directions. Now, setting $d = 26$, we have;

$$Z_{C_2} = iV_{26}\, n^2 \int_0^\infty \frac{dt}{2t} \left(8\pi^2\alpha' t\right)^{-13} \eta(it)^{-24}$$

We now want to study extreme t limits. $t \to \infty$ limit is like $\tau_2 \to \infty$ limit (**expand on this**). To study the $t \to 0$ limit, we do a modular transformation. Define $s = \pi/t$, and we get;

$$\eta\left(\frac{i}{t}\right) = \sqrt{t}\eta(it) \Rightarrow \eta\left(\frac{is}{\pi}\right) = \sqrt{\frac{\pi}{s}}\eta\left(\frac{i\pi}{s}\right)$$

Then, the partition function becomes;

$$-iV_{26}\int_0^\infty \frac{d(\pi/s)}{2(\pi/s)}\left(\frac{8\pi^3\alpha'}{s}\right)^{-13}\eta(i\pi/s)^{-24} = -iV_{26}\int_0^\infty \frac{d(\pi/s)}{2(\pi/s)}\left(\frac{8\pi^3\alpha'}{s}\right)^{-13}\left(\frac{\pi}{s}\right)^{12}\eta(is/\pi)^{-24}$$

$$= \frac{iV_{26}n^2}{2\pi(8\pi^2\alpha')^{13}}\int_0^\infty ds\eta(is/\pi)$$

Using the expansion of $\eta(\tau)$ i.e.;

$$\eta(\tau) = q^{1/24}(1 - q - q^2 + ...)$$

we have;

$$\eta(\tau)^{-24} = q^{-1}(1-q+...)^{-24} = e^{-2\pi i\tau}(1+24q+...) \Rightarrow \eta(is/\pi)^{-24} = e^{2s}(1+24e^{-2s}+...) = e^{2s}+24+\mathcal{O}(e^{-2s}) \quad (7.4.2)$$

So, we have divergence from the massless states as well. The massless divergent part is as follows;

$$2\frac{24iV_{26}n^2}{4\pi(8\pi^2\alpha')^{13}}\int_0^\infty ds \quad (7.4.3)$$

Include discussion about IR divergence, disk tadpole source term made out of dilaton and graviton (which modifies field equation), and boundary states.

7.4.2 The Klein bottle

The Klein bottle partition becomes;

$$\int_0^\infty \frac{dt}{4t}\operatorname{Tr}'_c\left[\Omega\exp(-2\pi t(L_0 + \tilde{L}_0))\right]$$

where 1/2 comes from the $\mathbb{Z}_2$ symmetry of K_2 (**think about this**) and another 1/2 comes due to the projection operator (**Think about this as well... a problem is here**). Using the expression for L_0 and $\bar{L}_0$ and noticing that the momentum integration is going to be like the torus case, we get;

$$iV_d\int_0^\infty \frac{dt}{4t}(4\pi al't)^{-d/2}\sum_{i\in\mathcal{H}_\perp}\Omega_i\exp\left[-2\pi t(h_i + \bar{h}_i - 2)\right]$$

We see that we have to sum over the states with a definite value of Ω_i i.e. the diagonal states of Ω. The diagonal states of Ω_i should have left and right-handed states in the same state (**Not totally convinced, find more**), and thus, we won't be summing over two different occupation numbers now as we did in the torus case but on one occupation number only (like the cylinder case). The difference from the cylinder case is that we will have an argument for the exponential appearing in the summation that is double the cylinder case and thus, we won't have $\eta(i\theta)$ now but $\eta(2it)$ (**Include steps if you want**). So, we would have;

$$Z_{K_2} = iV_{26}\int_0^\infty \frac{dt}{4t}(4\pi\alpha't)^{-13}\eta(2it)^{-24}$$

Include the rise of cross-cap here. For the modular transformations, we do the variable change to $s = \pi/2t$ and then =, we get;

$$Z_{K_2} = \frac{iV_{26}}{4}\int_0^\infty \frac{ds}{s}(2\pi\alpha')^{-13}s^{13}\eta\left(\frac{i\pi}{s}\right)^{-24}\pi^{-13} = \frac{iV_{26}}{4\pi(2\pi\alpha')^{13}}\int_0^\infty ds\,\eta(is/\pi)^{-24}$$

$$= \frac{iV_{26}2^{26}}{4\pi(8\pi\alpha')^{13}}\int_0^\infty ds\,\eta(is/\pi)^{-24}$$

where we used the fact;

$$\eta(i\pi/s) = \sqrt{\frac{s}{\pi}}\eta(is/\pi)$$

We can extract the massless divergent part from this partition function just like the cylinder case by expanding $\eta(is/\pi)$ and we saw from the cylinder case that the relevant term from this expansion is 24 and thus, the massless divergent part of Z_{K_2} is;

$$\frac{24iV_{26}2^{26}}{4\pi(8\pi\alpha')^{13}}\int_0^\infty ds \tag{7.4.4}$$

7.4.3 The Mobius strip

The Mobius strip partition function will have the same form as the cylinder (because all the arguments of the cylinder apply). However, it will have an additional 1/2 due to the projection operator (**problem here again**). So the required partition function is a s follows;

$$Z_{M_2} = iV_{26}\int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-13}\sum_{i\in\mathcal{H}_o^\perp}\Omega_i e^{-2\pi t(h_i-1)} = iV_{26}\int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-13}e^{2\pi t}\sum_{i\in\mathcal{H}_o^\perp}\Omega_i e^{-2\pi t N_i}$$

where N_i is the level of i−th state and $\mathcal{H}_o^\perp$ is the transverse Hilbert space for the open strings. Now, recall that;

$$\Omega\alpha_n^\mu\Omega^{-1} = (-1)^n\alpha_n^\mu$$

and thus, we can find the trace as follows;

$$e^{2\pi t}\prod_{\mu,N}\sum_{N_{\mu n}=0}^\infty(-1)^{nN_{\mu n}}e^{-2\pi t n N_{\mu n}} = e^{2\pi t}\prod_{\mu,N}\frac{1}{1-(-1)^n e^{-2\pi t n}} = e^{2\pi t}\prod_{n=1}^\infty\left[1-(-1)^n e^{-2\pi t n}\right]^{-24}$$

where $N_{\mu n}$ is the occupation number again. **(The expression for it not coming to be what Polchinski writes)**. This expression can be written in terms of $\vartheta_{00}(0,2it)$ as follows;

$$e^{2\pi t}\prod_{n=1}^\infty\left[1-(-1)^n e^{-2\pi t n}\right]^{-24} = \vartheta_{00}(0,2it)^{-12}\eta^{-12}(2it)$$

A digression here. All the states are constrained to have $\Omega^2 = 1$ but as we saw in chapter 6, the total eigenvalue of $\Omega\,|N,ij\rangle$ (where i,j are $SO(n)$ indices) which we call ω, is made up of two parts;

$$\omega = \omega_N s_a,\ \ \omega_N = (-1)^N$$

where s_a is +1 when λ_{ij}^a is symmetric and −1 when λ_{ij}^a is anti-symmetric. So, at even mass levels, we should have λ_{ij}^a anti-symmetric and vice versa. However, in the calculation of the partition function above, we included ω_N part of ω only. We can now include s_a part as well. The symmetric matrices contribute with $s_a = 1$ and vice versa. For the $Sp(k) = Sp(n/2)$ case (where $n = 2k$ and where $Sp(k)$ contains $2k\times 2k$ matrices), this flips and we have the symmetric matrices contributing with $s_a = -1$ and vice versa. All of this is discussed in chapter 6.

For $n\times n$ matrices, we have $n(n+1)/2$ symmetric matrices and $n(n-1)/2$ anti-symmetric states. So, the Chan Paton part of the partition function gives us the following factors;

$$SO(n): \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n$$

$$Sp(k): -\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = -n$$

So, the total partition function becomes;

$$Z_{M_2} = \pm inV_{26}\int_0^\infty\frac{dt}{4t}(8\pi^2\alpha' t)^{-13}\vartheta_{00}(0,2it)^{-12}\eta(2it)^{-12}$$

where the upper sign is for $SO(n)$ and lower sign is for $Sp(k)$. We now do the modular transformation of this partition function. We set $s = \pi/4t$ and then, we get;

$$Z_{M_2} = \pm\frac{inV_{26}}{4(2\pi^3\alpha')^{13}}\int_0^\infty ds\ \ s^{12}\vartheta_{00}\left(0,\frac{i\pi}{2s}\right)^{-12}\eta\left(\frac{i\pi}{2s}\right)^{-12}$$

Using the modular properties of $\vartheta_{00}(\nu,\tau)$ and $\eta(\tau)$ by setting $\nu = 0$ and $\tau = 2is/\pi$, we get the following;

$$\vartheta_{00}\left(0,\frac{i\pi}{2s}\right) = \sqrt{\frac{2s}{\pi}}\vartheta_{00}\left(0,\frac{2is}{\pi}\right), \quad \eta\left(\frac{i\pi}{2s}\right) = \sqrt{\frac{2s}{\pi}}\eta\left(\frac{2is}{\pi}\right)$$

and thus, the partition function becomes;

$$Z_{M_2} = \pm\frac{inV_{26}}{4(2\pi^3\alpha')^{13}}\frac{\pi^{12}}{2^{12}}\int_0^\infty ds\ \vartheta_{00}\left(0,\frac{2is}{\pi}\right)^{-12}\eta\left(\frac{2is}{\pi}\right)^{-12} = \pm 2\frac{inV_{26}2^{13}}{4\pi(8\pi^2\alpha')^{13}}\int_0^\infty ds\, \vartheta_{00}\left(0,\frac{2is}{\pi}\right)^{-12}\eta\left(\frac{2is}{\pi}\right)^{-12}$$

We now calculate the massless divergent part of this partition function. Using the expansion for η, we have;

$$\eta(\tau) = q^{1/24}(1-q-...) = e^{\pi i\tau/12}\left(1-e^{2\pi i\tau}-...\right) \Rightarrow \eta(2is/\pi) = e^{-s/6}\left(1-e^{-4s}-...\right)$$

$$\Rightarrow \eta(2is/\pi)^{-12} = e^{2s}(1-e^{-4s}-...)^{-12} = e^{2s}+12e^{-2s}-...$$

Moreover, from the expansion of $\vartheta_{00}(0,\tau)$, we have;

$$\vartheta_{00}(0,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = 1+2\sum_{n=1}^{\infty} e^{\pi i n^2\tau} = 1+2e^{\pi i\tau}+2e^{8i\pi\tau}+...$$

$$\Rightarrow \vartheta_{00}(0,2is/\pi) = 1+2e^{-2s}+... \Rightarrow \vartheta_{00}(0,2is/\pi) = (1+2e^{-2s}+...)^{-12} = 1-24e^{-2s}+...$$

Therefore, we have;

$$\vartheta_{00}(0,2is/\pi)^{-12}\eta(2is/\pi)^{-12} = \left(e^{2s}+12e^{-2s}-...\right)\left(1-24e^{-2s}+...\right) = e^{2s}-24+...$$

So, the massless divergent part from Z_{M_2} is as follows;

$$\mp 2\frac{24inV_{26}2^{13}}{4\pi(8\pi^2\alpha')^{13}}\int_0^\infty ds \tag{7.4.5}$$

Now, we can add all the massless divergent terms (which are also known as tadpole divergences) from C_2, K_2 and M_2 (recall that there is no massless divergent term from T^2). For the cylinder case, there will be an additional factor of 1/2 in (7.4.3) because of the projection operator. Adding (7.4.4), (7.4.5) and half of (7.4.3), we get;

$$\frac{24iV_{26}}{4\pi(8\pi^2\alpha')^{13}}\left(\underbrace{n^2}_{\text{cylinder}}\ \underbrace{\mp 2.2^{13}n}_{\text{Mobius strip}} + \underbrace{2^{26}}_{\text{Klein bottle}}\right)\int_0^\infty ds = \frac{24iV_{26}}{4\pi(8\pi^2\alpha')^{13}}(2^{13}\mp n)^2\int_0^\infty ds$$

We see that this term doesn't vanish for the plus sign, which is the lower sign, and if we recall that the lower sign was for the $Sp(k) = Sp(n/2)$ group, we see that this group isn't free of tadpole divergences. For the minus sign, this divergence cancels for $n = 2^{13} = 8192$. The upper sign is for the $SO(n)$ group; thus, the divergence vanishes only for $SO(8192)$.

8 Chapter 8: Toroidal compactification and T-duality

8.1 Toroidal compactification in field theory

We consider a $D = d + 1$ dimensional theory with one dimension compactified $x_d \sim x_d + 2\pi R$. The five-dimensional metric is G^D_{MN} and we adopt a particular convention for the metric as follows;

$$ds^2 = G^D_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\mu + G_{dd}(dx^d + A_\mu dx^\mu)^2 \tag{8.1.1}$$

Comparing the two-line elements, we have the following correspondences;

$$G_{dd} = G^D_{dd},\ G_{\mu d} = \frac{G^D_{\mu d}}{A_\mu},\ G_{\mu\nu} = G^D_{\mu\nu} - G^D_{dd} A_\mu A_\nu$$

All the fields depend on non-compact coordinates only and thus, the metric (8.1.1) is invariant in x^d translations and it is also manifestly invariant in $x'^\mu(x^\nu)$ reparametrizations. The expanded form of the metric is as follows;

$$(G_{\mu\nu} + G_{dd} A_\mu A_\nu) dx^\mu dx^\nu + G_{dd} dx^d dx^d + 2 G_{dd} A_\mu dx^d dx^\mu$$

This metric is invariant under the reparametrizations of the form (**derive this**);

$$x^d \to x^d + \lambda(x^\mu)$$

if A_μ's transform as follows;

$$A_\mu \to A_\mu - \partial_\mu \lambda$$

If we include x^d dependence, then the fourier series for a scalar field $\phi(x^M)$ is written as follows;

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) e^{inx^d/R}$$

and then, the D dimensional wave equation becomes;

$$\partial_M \partial^M \phi(x^M) = 0 \Rightarrow \phi(x^M) = \sum_{n\in\mathbb{Z}} \partial_\mu \partial^\mu \phi_n(x^\mu) - \frac{n^2 G^{dd}}{R^2} \phi_n(x^\mu) e^{inx^d/R} = 0$$

$$\Rightarrow \partial_\mu \partial^\mu \phi_n - \frac{n^2 G^{dd}}{R^2} \phi_n = 0 \Rightarrow \partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2} \phi_n = 0$$

where in the last step, we set $G_{dd} = 1$ for simplicity. Comparing this equation with the Klein-Gordon equation, we see that ϕ_n has the mass of n^2/R^2. This is called the Kaluza-Klein tower. **(Write the massless effective action and the relations between couplings)**.

8.2 Toroidal compactification in CFT

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8.2.1 Partition function

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8.2.2 Vertex operators

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8.2.3 A technicality

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8.2.4 DDF operators

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8.3 Closed strings and T duality

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8.3.1 Enhanced gauge symmetries

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8.3.2 Scales and couplings

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8.3.3 Higgs mechanism

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8.3.4 T duality

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8.4 Compactification of several dimensions

If we compactify k dimensions, we have;

$$X^m \sim X^m + 2\pi R, \;\; 26 - k \leq m \leq 25$$

8.4.1 The string spectrum

Then the internal metric is G_{mn} and we have a B field as B_{mn}. We also have $A_{\mu m}$ bosons (from G_{MN}) and $B_{\mu m}$ bosons (from B_{MN}). We do the analysis for zero modes. We take the zero modes as follows;

$$X^m(\sigma^1, \sigma^2) = x^m(\sigma^2) + w^m \sigma^1$$

The worldsheet lagrangian is as follows;

$$L = \frac{1}{2\pi}(g^{ab}G_{mn} + i\epsilon^{ab}B_{mn})\partial_a X^m \partial_b X^n$$

$$= \frac{1}{2\alpha'}(g^{22}G_{mn}\partial_2 X^m \partial_2 X^n + g^{11}G_{mn}\partial_1 X^m \partial_1 X^n + i\epsilon^{12}G_{mn}\partial_1 X^m \partial_2 X^n + i\epsilon^{21}G_{mn}\partial_2 X^m \partial_1 X^n)$$

$$= \frac{1}{2\alpha'}G_{mn}(\dot{x}^m \dot{x}^n + w^n w^m R^2) - \frac{i}{\alpha'}B_{mn}w^m \dot{x}^n R$$

The conjugate momentum p_m is given as;

$$p_m = i\frac{\partial L}{\partial \dot{x}^m} = \frac{1}{\alpha'}v_m - \frac{1}{\alpha'}B_{mn}w^m R \Rightarrow v_m = \alpha' \frac{n_m}{R} - B_{mn}w^n R$$

where $v_m = i\dot{x}_m$. Define p_{mL} and p_{mR} as follows;

$$p_{mL} = \frac{n_m}{R} + (\delta_{mn} - B_{mn})\frac{w^n R}{\alpha'} = \frac{v_{Lm}}{\alpha'}, \;\; p_{mR} = \frac{n_m}{R} - (\delta_{mn} + B_{mn})\frac{w^n R}{\alpha'} = \frac{v_{Rm}}{\alpha'}$$

where

$$v_L^m = v^m + w^m R, \; v_R^m = v^m - w^m R$$

This gives the following zero Virasoro operators;

$$L_0 = \frac{\alpha' p_L^2}{4} + N, \;\; \tilde{L}_0 = \frac{\alpha' p_R^2}{4} + \tilde{N}$$

The zero mode hamiltonian (**derive this**);

$$H = \frac{1}{2\alpha'}G_{mn}(v^m v^n + w^m w^n R^2) = \frac{1}{4\alpha'}G_{mn}(v_L^m v_L^m + v_R^m v_R^m)$$

Therefore, the mass in uncompactified dimensions is (**motivated but think more about it**);

$$m^2 = \frac{1}{2\alpha'^2} G_{mn}(v_L^m v_L^m + v_R^m v_R^m) + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

Moreover, from the $L_0, \tilde{L}_0$ modes above, we can compute the difference constraint as follows;

$$L_0 - \tilde{L}_0 = 0 \Rightarrow G_{mn}(v_L^m v_L^n - v_R^m v_R^n) + 4\alpha'(N - \tilde{N}) = 0 \Rightarrow n^m w_m + N - \tilde{N} = 0$$

(Write about the torus partition function argument). Define the alternative momenta k_{rL} and k_{rR} and coordinates X_L^r, X_R^r as follows;

$$G_{mn} = e_m^r e_n^r, \;\; X^r = e_m^r X^m, \; k_{rL} = e_r^m \frac{v_{mL}}{\alpha'}, \; k_{rR} = e_r^m \frac{v_{mR}}{\alpha'}$$

then the mass expression and the difference constraint become;

$$m^2 = \frac{1}{2}(k_{rL}k_{rL} + k_{rR}k_{rR}) + \frac{2}{\alpha'}(N + \tilde{N} - 2), \;\; \alpha'(k_{rL}k_{rL} - k_{rR}k_{rR}) + 4(N - \tilde{N}) = 0 \qquad (8.4.1)$$

8.4.2 Narain compactification

For Narain compactifications, define dimensionless momenta as follows;

$$l_{L,R} = \sqrt{\frac{\alpha'}{2}} k_{L,R}$$

For single valuedness of the OPE;

$$:e^{ik_L X_L(z) + ik_R X_R(\bar{z})} :: e^{ik'_L X_L(0) + ik'_R X_R(0)} :\sim z^{\alpha' k_L k'_L/2} \bar{z}^{\alpha' k_R k'_R/2} : e^{i(k+k')_L X_L(0) + i(k+k')_R X_R(0)} :$$
$$= z^{l_L l'_L} \bar{z}^{l_R l'_R} : e^{i(k+k')_L X_L(z) + i(k+k')_R X_R(z)} :$$

we need the following phase to be unity;

$$\exp[2\pi i(l_L l'_L - l_R l'_R)] = 1 \Rightarrow l_L l'_L - l_R l'_R \in \mathbb{Z} \Rightarrow l \circ l' \in \mathbb{Z}$$

where a new dot product has been defined with signature (k, k). So, the lattice of l's i.e. Γ is in its self dual lattice Γ^* i.e.

$$\Gamma \subset \Gamma^*$$

The partition function for a CFT (with $c = \bar{c}$) on the torus is given as follows;

$$\sum_{|\psi\rangle} \langle\psi| q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} |\psi\rangle = \sum_{|\psi\rangle} \langle\psi| e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \frac{c}{24})} |\psi\rangle$$

and it acquires the following phase under $\tau \to \tau + 1$;

$$2\pi i(L_0 - \tilde{L}_0)$$

and thus, for single valuedness, we need $L_0 - \tilde{L}_0 \in \mathbb{Z}$. Using the expressions above, we get;

$$L_0 - \tilde{L}_0 = \frac{1}{2}(l_{Lr}l_{Lr} - l_{Rr}l_{Rr}) + N - \tilde{N} \in \mathbb{Z} \Rightarrow l_{Lr}l_{Lr} - l_{Rr}l_{Rr} \in 2\mathbb{Z} \Rightarrow l \circ l \in 2\mathbb{Z}$$

From this condition, the singlevaluedness of the OPE can be derived;

$$(l + l') \circ (l + l') - l \circ l - l' \circ l' \in 2\mathbb{Z} \Rightarrow l \circ l' \in \mathbb{Z}$$

The partition function for mth direction with possible vaccua $|0; l_{rL}, l_{rR}\rangle$ is as follows;

$$Z_m = \frac{1}{|\eta(\tau)|^2} \sum_{l_L^m, l_R^m} q^{\frac{1}{2} l_L^{m2}} \bar{q}^{\frac{1}{2} l_R^{m2}} = \frac{1}{|\eta(\tau)|^2} \sum_{l_L^m, l_R^m} \exp\left(\pi i\tau l_L^{m2} - \pi i\bar{\tau} l_R^{m2}\right)$$

which means that the partition function for all m directions is as follows;

$$Z_\Gamma = \frac{1}{|\eta(\tau)|^{2k}} \sum_{l\in\Gamma} \exp\left(\pi i\tau l_L^2 - \pi i\bar{\tau} l_R^2\right)$$

The S transformation is as follows;

$$Z_\Gamma = \frac{1}{|\eta(\tau)|^{2k}} \sum_{l'\in\Gamma} \int d^{2k}l\,\delta(l-l') \exp\left(\pi i\tau l_L^2 - \pi i\bar{\tau} l_R^2\right) = V_\Gamma^{-1} \frac{1}{|\eta(\tau)|^{2k}} \sum_{l''\in\Gamma} \int d^{2k}l \exp\left(2\pi i l''\circ l + \pi i\tau l_L^2 - \pi i\bar{\tau} l_R^2\right)$$

where we used the following identity (**derive this**);

$$\sum_{l'\in\Gamma} \delta(l-l') = V_\Gamma^{-1} \sum_{l''\in\Gamma^*} \exp\left(2\pi i l''\circ l\right)$$

Using Poisson resummation formula and the modular transformation of η function, we get;

$$Z_\Gamma(\tau) = V_\Gamma^{-1} Z_{\Gamma^*}(-1/\tau)$$

Then, $\Gamma = \Gamma^*$ is sufficient to ensure modular invariance. **(Why is it necessary if modular invariance is true for all τ ?)** Thus Γ has to be an even, self dual lattice. If $\Lambda \in O(k,k,\mathbb{R})$, then Γ is still an even, self dual lattice because the modular invariance conditions depend on $\circ$ dot product only. But $O(k,k,\mathbb{R})$ is not a symmetry because the expression for mass and the difference constraint aren't invariant under it. They are invariant only in $O(k,\mathbb{R})\times O(k,\mathbb{R})$. Thus, the group that generates inequivalent theories (e.g. theories on lattices of different radii) is as follows;

$$\frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R})\times O(k,\mathbb{R})}$$

and the dimension of this group is;

$$\frac{2k(2k-1)}{2} - 2\times\frac{k(k-1)}{2} = k(2k-1) - k(k-1) = k^2$$

which is the same as the DOF from G_{mn} and B_{mn}. We can start from a lattice (known as Γ_0) with all compact dimensions compact and all radii being self dual radii. If we denote the group that maps Γ_0 to itself as $O(k,k,\mathbb{Z})$, then the following lattices are the same;

$$\Lambda'\Lambda\Lambda''\Gamma_0 = \Lambda\Gamma,\ \ \Lambda\in O(k,k,\mathbb{R}),\ \Lambda''\in O(k,k,\mathbb{Z}),\ \ \Lambda'\in O(k,\mathbb{R})\times O(k,\mathbb{R})$$

Recalling the expressions for l_L^r and l_R^r;

$$l_L^r = \frac{n^r}{R} + (\delta^{rs} - B^{rs})\frac{\omega_s R}{\alpha'},\ l_R^r = \frac{n^r}{R} - (\delta^{rs} + B^{rs})\frac{\omega_s R}{\alpha'}$$

we see that Z_Γ is invariant if we do the following T duality transformation (**what about n^r, w^r transformations?**);

$$R \longleftrightarrow \frac{\alpha'}{R}$$

We see that if x^m is periodic with period $2\pi R$ then so is the combination;

$$x^m = L_n^m x^n,\ L_n^m \in \mathbb{Z}$$

but if $\det L \neq 1$, then the volume of the k torus changes and hence, we should have $\det L = 1$. Thus, $L \in SL(2,\mathbb{Z})$. Another component in the T duality transformations is the following;

$$b_{mn} \to b_{mn} + N_{mn} \Rightarrow B_{mn} \to B_{mn} + \frac{\alpha' N_{mn}}{R^2},\ \ N_{mn}\in\mathbb{Z}$$

This transformation does the following;

$$v_m^L = \frac{n_m}{R} + (\delta_{mn} - B_{mn})w^n R \to \frac{(n_m - N_{mn}w^n)}{R} + (\delta_{mn} - B_{mn})w^n R = \frac{n'_m}{R} + (\delta_{mn} - B_{mn})w^n R,\ \ n'_m = n_m - N_{mn}w^n \in \mathbb{Z}$$

and similarly, for v_m^R. So, we see that the partition function doesn't change because we are still summing over integer n^m and w^m.

8.4.3 An example

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8.5 Orbifolds

Let's start with a $\mathbb{Z}_2$ orbifold. The orbifold action is;

$$X^{25} \to -X^{25} \tag{8.5.1}$$

which sends all the operators appearing in X^{25} to their negatives. A compact space is made by an additional identification;

$$X^{25} \to -X^{25},\ X^{25} \sim X^{25} + 2\pi R m,\ m \in \mathbb{Z} \tag{8.5.2}$$

This implies that there is a new sector in the game which is called the twisted sector. It is given as follows;

$$X^{25}(\sigma + 2\pi) \sim -X^{25}(\sigma) \tag{8.5.3}$$

Since X^{25} is anto periodic, we have the following expansion for it;

$$X^{25}(z,\bar{z}) = i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{1}{m+1/2}\left(\frac{\alpha_m^{25}}{z^{m+1/2}} + \frac{\tilde{\alpha}_m^{25}}{\bar{z}^{m+1/2}}\right) \tag{8.5.4}$$

For a compact orbifold, there is another twister sector as follows;

$$X^{25}(\sigma + 2\pi) \sim 2\pi R - X^{25}(\sigma) \tag{8.5.5}$$

For this sector, we have the following expansion;

$$X^{25}(z,\bar{z}) = \pi R + i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{1}{m+1/2}\left(\frac{\alpha_m^{25}}{z^{m+1/2}} + \frac{\tilde{\alpha}_m^{25}}{\bar{z}^{m+1/2}}\right) \tag{8.5.6}$$

Now, we talk about the allowed massless states. For the generic massless states, we have $n = w = 0$ and thus, for the states to be invariant under the orbifold action, we should require that the number of 25 direction creation operators in the state is even. This is violated only by the Kaluza Klein gauge bosons and the gauge bosons from the anti-symmetric tensor. Thus, those states are ruled out. The zero point energy constant for the twisted sector is determined as follows now (we have 23 periodic and 1 anti-periodic boson);

$$a_{\text{twisted}} = -\frac{23}{24} + \frac{1}{48} = -\frac{15}{16} \tag{8.5.7}$$

Then, the mass-shell condition becomes as follows;

$$\frac{\alpha' p^2}{4} + N + a_{\text{twisted}} = \frac{\alpha' p^2}{4} + N - \frac{15}{16} = 0 \Rightarrow m^2 = \frac{4}{\alpha'}\left(N - \frac{15}{16}\right) \tag{8.5.8}$$

Thus, the lowest mass state is tachyonic ($N = 0$). The level matching condition requires $N = \tilde{N}$ and due to the anti periodicity, X^{25} oscillators make half integral contributions to the number operator. Thus, the next excited state involves $\alpha_{-1/2}^{25}\tilde{\alpha}_{-1/2}^{25}$ and hence, the this state is also tachyonic. There is no way to set $N = 15/16$ and hence, there is no massless state.
Vertex operators for twisted states are hard to find (**find more about this**). Untwisted external states for a tree level amplitude only involves untwisted states. This is called the inheritance principle (**find more about this**). The partition function for the untwisted sector is given as follows (this is well known and thus, I won't give the details);

$$\text{Tr}\left(\frac{1+r}{2} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}}\right) = \frac{1}{2} Z_{\text{tor}}(R,\tau) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \tag{8.5.9}$$

where r in the orbifold action and ϑ_2 is given as follows;

$$\vartheta_2(\tau) = 2\eta(\tau) q^{\frac{1}{12}} \prod_{r\geq 1}(1+q^r)^2 = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau) = \vartheta_{10} \tag{8.5.10}$$

where I used another name for these functions that comes from a general formula for these functions (given later). Moreover, the last expression can be derived from the Jacobi triple product identity;

$$q^{-\frac{1}{24}}\prod_{r\geq 1}(1+q^{r+\frac{1}{2}}w)(1+q^{r+\frac{1}{2}}w^{-1}) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}w^n \tag{8.5.11}$$

The partition function in (8.5.9) is not modular invariant because it is not invariant in S transformations. We will have to add the twisted sector. The partition function for the twisted sector is as follows;

$$\mathrm{Tr}\left(\frac{1+r}{2}q^{L_0+\frac{1}{48}}\bar{q}^{\bar{L}_0+\frac{1}{48}}\right) = \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| \tag{8.5.12}$$

The factor of 1/48 comes because of the additional 1/16 term (that comes from the weight of the first excited state in the twisted sector. **Find more about this**). Moreover, the functions used are defined as follows;

$$\vartheta_3(\tau) = q^{-\frac{1}{24}}\eta(\tau)\prod_{r=0}^{\infty}(1+q^{r+\frac{1}{2}})^2 = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \vartheta_{00} \tag{8.5.13}$$

$$\vartheta_4(\tau) = q^{-\frac{1}{24}}\eta(\tau)\prod_{r=0}^{\infty}(1-q^{r+\frac{1}{2}})^2 = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}} = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau) = \vartheta_{01} \tag{8.5.14}$$

These functions are related to each other by modular transformations as follows;

$$\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \overset{T}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| \overset{S}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| \overset{T}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| \overset{S}{\leftrightarrow} \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| \tag{8.5.15}$$

The general formula for ϑ functions is as follows;

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2}e^{2\pi i(n+a)(z+b)} \tag{8.5.16}$$

for which a product representation can found as follows (**derive this**);

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \eta(\tau)e^{2\pi i a(z+b)}q^{\frac{a^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^{n+a-\frac{1}{2}}e^{2\pi i(z+b)}\right)\left(1+q^{n-a-\frac{1}{2}}e^{-2\pi i(z+b)}\right) \tag{8.5.17}$$

8.5.1 Twisting

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8.5.2 $c = 1$ CFTs

The moduli space of compact $c = 1$ CFTs contains the torus theories and the orbifold theories. These meet at a point because of the following; (**prove**);

$$Z_{\mathrm{orb}}(\sqrt{\alpha},\tau) = Z_{\mathrm{tor}}(2\sqrt{\alpha},\tau)$$

i.e. partition function of the torus theory at twice the self dual radius has the same partition function as the orbifold theory at the self dual radius. (**Write about the Z_k and D_k, T, O and I twists**).

8.6 Open strings

Let's start with a pure gauge $A_{25}(x^M) = 0$ which is gauge equivalent to the following (**the coupling constant is not included in there. Find out why**);

$$A_{25}(x^M) = -i\exp\left(\frac{i\theta x^{25}}{2\pi R}\right)\partial_{25}\exp\left(\frac{-i\theta x^{25}}{2\pi R}\right) = -\frac{\theta}{2\pi R}$$

where θ is constant. Now, the Wilson line for this potential is as follows;

$$W = \exp\left(iq\oint dx^{25}A_{25}\right) = \exp\left(-\frac{iq\theta}{2\pi R}\oint dx^{25}\right) = \exp\left(-\frac{iq\theta}{2\pi R}(2\pi R)\right) = \exp\left(-iq\theta\right) \tag{8.6.1}$$

where q is the coupling constant. Now, suppose that we have a point particle coupled to an electromagnetic field. The action is as follows (where τ is the Euclidean time parameter **-derive this-**);

$$S = \int d\tau \left(\frac{1}{2}\dot{X}^M \dot{X}_M + \frac{m^2}{2} - iqA_M \dot{X}^M \right)$$

The conjugate momentum p_{25} is as follows;

$$p^{25} = \frac{\partial L}{\partial(\partial X^{25}/\partial\tau_{\text{Minkowski}})} = -\frac{\partial L}{i\partial(\partial X^{25}/\partial\tau_{\text{Euclidean}})} = i(\dot{X}^{25} - iqA^{25}) = i\dot{X}^{25} + qA^{25} = v^{25} - \frac{q\theta}{2\pi R}$$

where $v^{25} = i\dot{X}^{25}$. Now, due to the periodicity, $p^{25} = l/R$, $l \in \mathbb{Z}$. and thus, we have;

$$v^{25} = \frac{l}{R} + \frac{q\theta}{2\pi R}$$

The hamiltonian is derived as follows (**derive this**);

$$H = \frac{1}{2}(p^\mu p_\mu + v_{25}^2 + m^2)$$

Since this hamiltonian annihilates the physical states, the mass-shell condition for the mass that is visible in non-compact dimensions becomes;

$$-p^\mu p_\mu = m^2 + v_{25}^2 = m^2 + \left(\frac{l}{R} + \frac{q\theta}{2\pi R} \right)^2$$

we see that if $R \to \infty$, then the mass doesn't get any correction. Now, we talk about the non abelian gauge gauge theory. Open string has Chan Paton indices for $U(n)$ on its endpoints. If we have a constant $A^a_{25}T^a$ where T^a are generators of $U(n)$, then since T^a is hermitian, we can use a constant gauge factor (which is unitary) to diagonalize $A^a_{25}T^a$ as follows;

$$A^a_{25}T^a = -\frac{1}{2\pi R}\text{diag}(\theta_1, ..., \theta_n)$$

For a general Chan paton state;

$$|N;k;a\rangle = \sum_{ij} |N;k;ij\rangle \lambda^a_{ij}$$

the guage field A_{25} couples as follows;

$$|N;k;a\rangle = \sum_{ij} |N;k;ij\rangle [A_{25}, \lambda^a]_{ij}$$

The required commutator is as follows;

$$[A_{25}, \lambda^a]_{ik} = \frac{1}{2\pi R} \sum_j (\theta_i \delta_{ij} \lambda_{jk} - \theta_j \lambda_{ij} \delta_{jk}) = (\theta_i - \theta_k)\lambda_{ik} \Rightarrow [A_{25}, \lambda^a]_{ij} = (\theta_i - \theta_j)\lambda_{ij}$$

So, we get see that the state $|N;k;ij\rangle$ has charge $+1$ in $U(1)_i$ and charge -1 in $U(1)_j$. So, we can deduce the expression for v_{25} in this case as follows;

$$v_{25} = \frac{l}{R} + \frac{\theta_i - \theta_j}{2\pi R}$$

Thus, mass of the string state $|N;k;ij\rangle$ is as follows;

$$m^2 = \frac{1}{\alpha'}(N-1) + \frac{(2\pi l + \theta_i - \theta_j)^2}{4\pi^2 R^2}$$

The lowest non-tachyonic states have the following mass;

$$m^2_{ij} = \frac{(\theta_i - \theta_j)^2}{4\pi^2 R^2}, \quad l = 0, N = 1 \tag{8.6.2}$$

These states are massive if all the θ's are different. If θ's are equal in s sets of size r_i, then the massless states form the following representation;

$$U(n) \to U(r_1) \times ... \times U(r_s)$$

8.6.1 T duality

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8.7 D Branes

The gauge field in the compact direction dictates the translation of the D brane in the dual compact direction and thus, it dictates the shape of the D brane. T dualizing a tangent direction reduces the dimension of the D brane and vice versa. Coincident r branes will give the following massless states;

$$\alpha^{\mu}_{-1}|k;ij\rangle \; i,j \in \{1,...,r\}$$

and thus, we have r^2 massless vectors and a $U(r)$ symmetry. We also have r^2 massless scalars as follows;

$$\alpha^{25}_{-1}|k;ij\rangle \; i,j \in \{1,...,r\}$$

8.7.1 The D-brane action

We now try to guess the low-energy D-brane action. This action is called the DBI action.For a $p-$brane, we introduce the coordinates $\xi^0,...\xi^p$ on the brane world volume. The fields to consider on the brane are the embedding fields $X^{\mu}(\xi)$ and the gauge fields $A_a(\xi)$ (recall that gauge fields live on the endpoints of the strings). Now, we define the induced metric $G_{ab}(\xi)$ and induced $B_{ab}(\xi)$ field as follows;

$$G_{ab}(\xi) = \frac{\partial X^{\mu}}{\partial X^a}\frac{\partial X^{\nu}}{\partial X^b}G_{\mu\nu}(X(\xi)), \; B_{ab}(\xi) = \frac{\partial X^{\mu}}{\partial X^a}\frac{\partial X^{\nu}}{\partial X^b}B_{\mu\nu}(X(\xi))$$

Now, if we look at the Nambu-Goto action, the obvious generalization of the integrand would be $\sqrt{-\left[\det\left(G_{ab}+B_{ab}\right)\right]}$ with an overall factor of $-T_p$ i.e. the brane tension. (**show that this determinant will give the factor of i in the Polyakov type action for D-Branes**). However, this is not the whole story. Since we are trying to guess the open string tree level action, we want to have g_o^{-2} in the action. However, $g_o^{-2} \sim e^{-\phi}$ in the lagrangian. Now, we prove that B_{ab} should appear only with F_{ab}. In string worldsheet action, we have the following combination;

$$\frac{i}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}B_{\mu\nu} + i\int_{\partial M} dX^{\mu}A_{\mu}$$

where $\delta A_{\mu} = \partial_{\mu}\lambda$ is a guage symmetry of this action. The gauge transformation of $B_{\mu\nu}$ is as follows;

$$B_{\mu\nu} = \partial_{\mu}\zeta_{\nu} - \partial_{\nu}\zeta_{\mu}$$

Under this transformation, the action changes as follows;

$$\frac{i}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\left(\partial_{\mu}\zeta_{\nu} - \partial_{\nu}\zeta_{\mu}\right) = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\partial_{\mu}\zeta_{\nu}$$

$$= \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left(\partial_1 X^{\mu}\partial_2 X^{\nu} - \partial_2 X^{\mu}\partial_1 X^{\nu}\right)\partial_{\mu}\zeta_{\nu} = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_1 X^{\mu}\partial_2 X^{\nu}\left(\partial_{\mu}\zeta_{\nu} - \partial_{\nu}\zeta_{\mu}\right)$$

We now use the following identities;

$$\partial_1\zeta_{\mu} = \partial_{\nu}\zeta_{\mu}\partial_1 X^{\nu}, \;\; \partial_2\zeta_{\mu} = \partial_{\nu}\zeta_{\mu}\partial_2 X^{\nu}$$

which imply;

$$\partial_1 X^{\mu}\partial_2 X^{\nu}\left(\partial_{\mu}\zeta_{\nu} - \partial_{\nu}\zeta_{\mu}\right) = \partial_2 X^{\nu}\partial_1\zeta_{\nu} - \partial_1 X^{\mu}\partial_2\zeta_{\mu}$$

and thus, the variation of the action becomes;

$$-\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left(\partial_1 X^{\mu}\partial_2\zeta_{\mu} - \partial_2 X^{\mu}\partial_1\zeta_{\mu}\right) \tag{8.7.1}$$

The first term becomes;

$$-\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_1\left(X^{\mu}\partial_2\zeta_{\mu}\right) + \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^{\mu}\partial_1\partial_2\zeta_{\mu}$$

$$= -\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}\,(X^\mu\partial_2\zeta_\mu)|_{\text{boundary}} + \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_1\partial_2\zeta_\mu \tag{8.7.2}$$

The second term in (8.7.1) becomes;

$$\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_2X^\mu\partial_1\zeta_\mu = \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}\partial_2\left(X^\mu\partial_1\zeta_\mu\right) - \frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_2\partial_1\zeta_\mu$$
$$= -\frac{i}{2\pi\alpha'}\int_M d^2\sigma\sqrt{g}X^\mu\partial_2\partial_1\zeta_\mu \tag{8.7.3}$$

where we used the fact that A_μ variation vanishes at starting and ending times. We also see that (8.7.3) cancels the second term in (8.7.2). So, the variation in the B term becomes;

$$-\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}\,(X^\mu\partial_2\zeta_\mu)|_{\text{boundary}} = -\frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}X^\mu\partial_2\zeta_\mu$$

where we have excluded the boundary label now because it is understood. This term can be canceled if A_μ also transforms as $\delta A_\mu = -\zeta_\mu/2\pi\alpha'$. This will give the following variation;

$$i\int_{\partial M} dX^\mu\delta A_\mu = i\int\int_{\partial M} d\sigma^2\sqrt{g}\partial_2X^\mu\delta A_\mu = i\int_{\partial M} d\sigma^2\sqrt{g}\partial_2(X^\mu\delta A_\mu) - \int d\sigma^2\sqrt{g}X^\mu\partial_2\delta A_\mu$$
$$= \frac{i}{2\pi\alpha'}\int_{\partial M} d\sigma^2\sqrt{g}X^\mu\partial_2\zeta_\mu$$

where we used the fact that δA_μ vanishes at the initial and final times (i.e. σ^2). This exactly cancels the variation of the B action. Now, the variation of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is as follows;

$$\delta F_{\mu\nu} = \partial_\mu\left(-\frac{\zeta_\nu}{2\pi\alpha'}\right) - \partial_\nu\left(-\frac{\zeta_\mu}{2\pi\alpha'}\right) = -\frac{1}{2\pi\alpha'}(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu) = -\frac{1}{2\pi\alpha'}\delta B_{\mu\nu} \Rightarrow \delta(B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) = 0$$

Thus, the only gauge invariant quantity is $B_{\mu\nu}+2\pi\alpha' F_{\mu\nu}$ and the induced field on the D-brane is $B_{ab}+2\pi\alpha' F_{ab}$. So, this combination should appear in the action. The DBI action is thus as follows;

$$S_{\text{DBI}} = -T_p\int d^{p+1}\xi e^{-\phi}[-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})]^{1/2} \tag{8.7.4}$$

For n separated branes, we have n copies of the action. For n coincident branes, the gauge field A_μ and X^μ become $n\times n$ matrices. The gauge field becomes $U(n)$ gauge field and the phenomenon of non-commutative geometry arises (**Find more about it and about the potential of the branes and its relation to the number of flat directions**).

8.7.2 D-Brane tension

We first derive the D-brane tension recursion relation. Since T_p doesn't depend on fields, we can derive the relation for static D-branes and that relationship should be the same for non-static branes. Firstly, consider a static D-brane i.e. the B field and gauge field is zero, the dilaton is constant and the metric is flat (**Is this description correct?**). Then, the action becomes;

$$S = -T_p\int d^{p+1}\xi e^{-\phi} = -T_pe^{-\phi}\int d^{p+1}\xi$$

Now, the action for a static brane has no kinetic energy, and the only potential energy is the mass of the brane. So, action is negative of the mass of the brane. Therefore, the mass per unit volume of the Dp-brane is $T_pe^{-\phi}$. Now, consider the case when Dp-brane is wrapped around a p-torus. The volume of that torus is;

$$\prod_{i=1}^{p}(2\pi R_i)$$

and thus, the mass of the Dp-brane is;

$$T_pe^{-\phi}\prod_{i=1}^{p}(2\pi R_i)$$

Now, let T duality act on p-th direction of the torus. We saw in section 8.3 that ϕ changes as follows under T-duality;

$$e^{\phi} \to e^{-\phi'} = \frac{R}{\sqrt{\alpha'}}e^{-\phi} \Rightarrow e^{-\phi} = \frac{\sqrt{\alpha'}}{R}e^{-\phi'}$$

Under T-duality, mass is invariant and thus, the mass of the brane is still given as;

$$T_p \frac{\sqrt{\alpha'}}{R_p}e^{-\phi'}\prod_{i=1}^{p}(2\pi R_i) = T_p\sqrt{\alpha'}e^{-\phi'}\prod_{i=1}^{p-1}(2\pi R_i)$$

But, this mass can also be written down as follows (because after T-duality, we have a D$(p-1)$ brane);

$$T_{p-1}e^{-\phi'}\prod_{i=1}^{p-1}(2\pi R_i)$$

and thus, we get;

$$2\pi\sqrt{\alpha'}T_p = T_{p-1} \Rightarrow T_p = \frac{T_{p-1}}{2\pi\sqrt{\alpha'}} \tag{8.7.5}$$

To acquire the expression of D brane tension, we can calculate the cylinder partition function of an open string stretched between two D branes. It is like the partition function calculated in chapter 7 but now, the string's endpoints can move only on the D branes, and thus, the momentum integration will be on $p+1$ directions and we will have V_{p+1} instead of V_{26}. Moreover, there will be an additional contribution to the conformal weight h_i for the $i-$th state (which we also called level N before but it would not be the case now) because the string is stretched between two D branes. Assume that the two D-branes are at $X^m = 0$ and $X^m = y^m$ for $m = p+1, ..., 25$. Now, we know that the expansion for X^m_L and X^m_R is as follows;

$$X^m_L = x^m_L + \alpha' p^m_L \sigma^+ + ..., \;\; X^m_R = x^m_R + \alpha' p^m_R \sigma^- + ...$$

$$\Rightarrow X^m(\tau,\sigma) = x^m_L + x^m_R + \alpha'(p^m_L + p^m_R)\tau + \alpha'(p^m_L - p^m_R)\sigma + ...$$

Imposing $X^m(0,\tau) = 0$ and $X^m(\pi,\tau) = y^m$ gives us the following;

$$p^m_L + p^m_R = 0, \;\; \alpha'(p^m_L - p^m_R) = y^m \Rightarrow p^m_L = -p^m_R = \frac{y^m}{2\alpha'\pi}$$

Now, we can deduce that;

$$L_0 = \alpha' p_L^2 + N = N + \frac{y^2}{4\pi^2\alpha'}$$

and thus, the additional contribution from stretching between the branes is $y^2/4\pi^2\alpha$. Now, looking at (7.4.1), we see that we have an additional factor;

$$\int_0^\infty \frac{dt}{2t}\mathrm{Tr}'_0\left[e^{-2\pi t L_0}\right] = \int_0^\infty \frac{dt}{2t}\mathrm{Tr}'_0\left[e^{-2\pi t\left(L_0^{\text{old}}+y^2/4\pi^2\alpha'\right)}\right] = \int_0^\infty \frac{dt}{2t}e^{-y^2 t/2\pi\alpha'}\mathrm{Tr}'_0\left[e^{-2\pi t L_0^{\text{old}}}\right]$$

So, the cylinder partition function is as follows;

$$\mathcal{A} = iV_{p+1}\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-(p+1)/2}\exp\left(-ty^2/2\pi\alpha'\right)\eta(it)^{-24} = \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty \frac{dt}{t^{(p+3)/2}}\exp\left(-ty^2/2\pi\alpha'\right)\eta(it)^{-24}$$

Now, we need to do this integral and for that, we need to expand $\eta(it)$ for large t and for that, we do the following steps;

$$\eta(it) = \frac{1}{\sqrt{t}}\eta(i/t) \Rightarrow \eta(it)^{-24} = t^{12}\eta(i/t)^{-24} = t^{12}\left(e^{2\pi/t} + 24 + ...\right)$$

using this expansion, the amplitude becomes;

$$\mathcal{A} = \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\;\; t^{(21-p)/3}\exp\left(-ty^2/2\pi\alpha'\right)\left(e^{2\pi/t} + 24 + ...\right)$$

We now need to evaluate the second term in this partition function. It is done as follows;

$$\frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\;\; t^{(21-p)/3}\exp\left(-ty^2/2\pi\alpha'\right) = \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{y^2}\right)^{(23-p)/2}\int_0^\infty d\mu\;\; \mu^{21-p}e^{-\mu}$$

$$= \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{y^2}\right)^{(23-p)/2}\Gamma\left(\frac{23-p}{2}\right) = iV_{p+1}\frac{24}{2^{12}}(4\pi\alpha')^{11-p}\pi^{(p-23)/2}\Gamma\left(\frac{23-p}{2}\right)|y|^{p-23}$$

where we did the variable change $\mu = ty^2/2\pi\alpha'$. Now, we use the result that Green's function for the laplacian $-\nabla^2$ is (**derive this**);

$$G_d(|y|) = \frac{1}{4\pi^{d/2}|y|^{d-2}}\Gamma\left(\frac{d-2}{2}\right) \tag{8.7.6}$$

and then, setting $d = 25 - p$, we get;

$$G_{25-p}(|y|) = \frac{1}{4\pi^{(25-p)/2}y^{23-p}}\Gamma\left(\frac{23-p}{2}\right) \Rightarrow 4\pi G_{25-p}(|y|) = \pi^{(p-23)/2}\Gamma\left(\frac{23-p}{2}\right)|y|^{p-23}$$

Therefore, we have;

$$\mathcal{A} = iV_{p+1}\frac{24\pi}{2^{10}}(4\pi\alpha')^{11-p}G_{25-p}(|y|)$$

We can convert this Green's function to the momentum space picture for the massless particles. Then, the amplitude becomes (**confirm this step**);

$$G_{25-p}(|y|) = \frac{1}{k_\perp^2} \Rightarrow \mathcal{A} = iV_{p+1}\frac{24\pi}{2^{10}k_\perp^2}(4\pi\alpha')^{11-p} \tag{8.7.7}$$

Now, we do the analogous field theory calculation because, from this calculation, we will find an expression for the required amplitude that will depend on T_p. We write the spacetime action for the strings with $D = 26$ but in the Einstein frame with $H_{\mu\nu\alpha}$ term excluded because the B field doesn't couple to the D brane (**Why? because of unoriented?**). The relevant terms are as follows;

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}}\left(\tilde{R} - \frac{1}{6}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\right)$$

where recall that $\tilde{\phi}$ has a zero vev because $\phi = \tilde{\phi} + \phi_0$ where $\phi_0 = \langle\phi\rangle$ Now, the relevant terms from the D brane action is (**Why no B field and gauge field**);

$$S = -T_p\int d^{p+1}\xi e^{-\phi}\sqrt{-\det G_{ab}} = -T_pe^{-\phi_0}\int d^{p+1}\xi e^{-\tilde{\phi}}\sqrt{-\det G_{ab}} = -\tau_p\int d^{p+1}\xi e^{-\tilde{\phi}}\sqrt{-\det G_{ab}}$$

where $\tau_p = T_pe^{-\phi_0}$ Now, we recall that the Einstein frame metric $\tilde{G}_{\mu\nu}$ and $G_{\mu\nu}$ are related as follows;

$$\tilde{G}_{\mu\nu} = e^{-4\tilde{\phi}/(D-2)}G_{\mu\nu} = e^{-\tilde{\phi}/6}G_{\mu\nu} \Rightarrow G_{\mu\nu} = e^{\tilde{\phi}/6}\tilde{G}_{\mu\nu} \Rightarrow \tilde{G}_{ab} = e^{\tilde{\phi}/6}G_{ab} \Rightarrow \det\tilde{G} = e^{(p+1)\tilde{\phi}/6}\det G$$

$$\Rightarrow \sqrt{-\det\tilde{G}} = e^{(p+1)\tilde{\phi}/12}\sqrt{-\det G}$$

Therefore, the D Brane action becomes;

$$S = -\tau_p\int d^{p+1}\xi\ \ e^{(p-11)\tilde{\phi}/12}\sqrt{-\tilde{G}}$$

In the linear approximation;

$$\tilde{G}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

while raising the indices by the flat metric, the spacetime action become (**derive them**);

$$S = -\frac{1}{8\kappa^2}\int d^{26}X\left(\partial_\mu h_{\nu\lambda}\partial^\mu h^{\nu\lambda} - \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{2}{3}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\right) \tag{8.7.8}$$

where $h = h^\mu_\mu$. The D-brane field action becomes;

$$S_p = -\tau_p\int d^{p+1}\xi\exp\left(\frac{p-11}{12}\tilde{\phi}\right)[-\det(\eta_{ab} + h_{ab})] = -\tau_p\int d^{p+1}\xi\left(1 + \frac{p-11}{12}\tilde{\phi}\right)\left(1 - \frac{1}{2}h_{aa}\right)$$

$$= -\tau_p\int d^{p+1}\xi - \tau_p\int d^{p+1}\xi\left[\frac{p-11}{12}\tilde{\phi} - \frac{h_{aa}}{2}\right] + \mathcal{O}(h\tilde{\phi}) \to -\tau_p\int d^{p+1}\xi\left[\frac{p-11}{12}\tilde{\phi} - \frac{h_{aa}}{2}\right] + \mathcal{O}(h\tilde{\phi}) \tag{8.7.9}$$

where we dropped the term that doesn't have fields in it. We also kept the indices on the trace in compact directions to differentiate it from h. **(Why not this linear term)**. From (8.7.8), we can read off the propagators. Look at the kinetic terms for $\tilde{\phi}$. We can define a new dilaton as;

$$\phi' = \frac{i}{\kappa\sqrt{6}}\tilde{\phi}$$

and then, the kinetic term for ϕ' becomes;

$$-\frac{1}{12\kappa^2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi'$$

i.e. the canonical kinetic term. The propagator for ϕ' is well known from QFT and thus, we have;

$$\langle\phi'\phi'\rangle = \frac{i}{k^2} \Rightarrow -\frac{1}{6\kappa^2}\langle\tilde{\phi}\tilde{\phi}\rangle = \frac{i}{k^2} \Rightarrow \langle\tilde{\phi}\tilde{\phi}\rangle = -\frac{6i\kappa^2}{k^2}$$

We see that for an arbitrary number of dimensions, we can define a dilaton as follows;

$$\phi' = \frac{2i}{\kappa\sqrt{D-2}}\tilde{\phi}$$

and ϕ' will still have the canonical kinetic term. So, for arbitrary number of dimensions, we have the following result.

$$\langle\phi'\phi'\rangle = \frac{i}{k^2} \Rightarrow -\frac{4}{(D-2)\kappa^2}\langle\tilde{\phi}\tilde{\phi}\rangle = \frac{i}{k^2} \Rightarrow \langle\tilde{\phi}\tilde{\phi}\rangle = -\frac{(D-2)i\kappa^2}{4k^2} \tag{8.7.10}$$

The kinetic term for $h_{\mu\nu}$ is as follows (**derive this**);

$$\langle h_{\mu\nu}h_{\alpha\beta}\rangle = -\frac{2i\kappa^2}{k^2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right) \tag{8.7.11}$$

The Feynman graph is read to give (**derive this**);

$$\mathcal{A} = \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[6\left(\frac{p-11}{12}\right)^2 + \frac{1}{2}\left(2(p+1) - \frac{1}{12}(p+1)^2\right)\right]$$

$$= \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[\frac{1}{24}(p-11)^2 + p + 1 - \frac{1}{24}(p+1)^2\right] = \frac{i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1}\left[\frac{1}{24}(p^2-22p+121) + p + 1 - \frac{1}{24}(p^2+2p+1)\right]$$

$$= \frac{6i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} \tag{8.7.12}$$

Now, comparing (8.7.7) and (8.7.12), we get;

$$\frac{6i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} = iV_{p+1}\frac{24\pi}{2^{10}k_\perp^2}(4\pi^2\alpha')^{11-p} \Rightarrow \tau_p^2 = \frac{\pi^2}{256\kappa^2}(4\pi^2\alpha')^{11-p} \Rightarrow \tau_p = \frac{\pi}{16\kappa}(4\pi^2\alpha')^{(11-p)/2}$$

$$\Rightarrow T_p e^{-\phi_0} = \frac{\pi}{16\kappa}(4\pi^2\alpha')^{(11-p)/2} \Rightarrow T_p = e^{\phi_0}\frac{\pi}{16\kappa}(4\pi^2\alpha')^{(11-p)/2} \tag{8.7.13}$$

Now, we can see that (8.7.13) satisfies (8.7.5) as follows;

$$T_{p-1} = e^{\phi_0}\frac{\pi}{16\kappa}(4\pi^2\alpha')^{(11-(p-1))/2} = e^{\phi_0}\frac{\pi}{16\kappa}(4\pi^2\alpha')^{(11-p)/2}(4\pi^2\alpha')^{1/2} = T_p 2\pi\sqrt{\alpha'} \Rightarrow T_p = \frac{T_{p-1}}{2\pi\sqrt{\alpha'}}$$

(Derive the relation between coupling constants).

8.8 T duality of unoriented string

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8.8.1 Open strings

9 Chapter 9: Higher order amplitudes

9.0.1 General tree level amplitudes

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9.0.2 Three point amplitudes

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9.0.3 Four point amplitudes and world-sheet duality

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9.0.4 Unitarity of the four point amplitude

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9.1 Higher genus Riemann surfaces

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9.1.1 Schottky groups

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9.1.2 Fuchsian groups

tt

9.1.3 The period matrix

tt

9.2 Sewing and cutting worldsheets

tt

9.2.1 A graphical decomposition

tt

9.3 Sewing and cutting CFTs

tt

9.3.1 General amplitudes

tt

9.3.2 String divergences

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9.4 String Field Theory

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9.5 Large order behaviour

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9.6 High energy and high temperature

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9.6.1 Hard scattering

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9.6.2 Regge scattering

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9.6.3 High temperature

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9.7 Low temperature and noncritical strings

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9.7.1 Non critical strings

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10 Chapter 10: Type I and type II strings

10.1 The superconformal algebra

In order to add spinors in the game, we need a two-dimensional analogue of the Dirac equation that is given as follows;

$$\rho^\alpha \partial_\alpha \psi^\mu = 0 \tag{10.1.1}$$

where $\alpha \in \{\tau, \sigma\}$ and ρ^α are two dimensional Dirac matrices. Now, the Dirac matrices are supposed to satisfy;

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \tag{10.1.2}$$

which means that $(\rho^0)^2 = -1, (\rho^0)^2 = 1$ and $\{\rho^0, \rho^1\} = 0$. This is satisfied if we take;

$$\rho^0 = -i\sigma^2,\ \rho^1 = \sigma^1 \tag{10.1.3}$$

Using the procedure described in the appendix B in volume 2 of Polchinski, we get the expression of the chiral matrix ρ as follows;

$$\rho^{0+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \rho^{0-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho = 2S_0 = 2\left(\rho^{0+}\rho^{0-} - \frac{1}{2}\mathbb{I}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the matrices in (10.1.3) are real, it is a Majorana representation and we can take the spinors to be real. We define $\bar{\psi}$ (for Majorana spinor) as follows;

$$\bar{\psi} = \psi^\dagger \rho^0 = \psi^T \rho^0 \tag{10.1.4}$$

The first equality is true for any spinor.
We can now start with the following action;

$$\begin{aligned} S = S_{\text{bosonic}} + S_{\text{fermionic}} &= \frac{1}{2\pi\alpha'}\int d^2z\, \partial X \bar{\partial} X + \frac{1}{4\pi}\int d^2z \left(\psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \bar{\partial}\tilde{\psi}_\mu\right) \\ &= \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X \bar{\partial} X + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right) \end{aligned} \tag{10.1.5}$$

The OPE's are as follows;

$$X^\mu(z) X^\nu(0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z|^2 \tag{10.1.6}$$

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z},\ \tilde{\psi}^\mu(z)\tilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{z} \tag{10.1.7}$$

which implies that;

$$\{\psi_0^\mu, \psi_0^\nu\} = \{\tilde{\psi}_0^\mu, \tilde{\psi}_0^\nu\} = \eta^{\mu\nu} \tag{10.1.8}$$

Now, we calculate the energy momentum tensor for (10.1.5). We do remind ourselves that;

$$g_{zz} = g_{\bar{z}\bar{z}} = 0,\ g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2z\bar{z}} \Rightarrow \sqrt{|g|} = \frac{1}{2z\bar{z}} \tag{10.1.9}$$

which implies that;

$$\partial^z = g^{z\bar{z}}\partial_{\bar{z}} = 2z\bar{z}\bar{\partial},\ \partial^{\bar{z}} = g^{\bar{z}z}\partial_z = 2z\bar{z}\partial \tag{10.1.10}$$

Now, we have (**not derived yet**);

$$\begin{aligned} -\frac{1}{2\pi}T^{\bar{z}}{}_z &= \frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}X^\mu)}\partial_z X^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}\psi^\mu)}\partial_z\psi^\mu - g_{zz}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\bar{\partial}X^\mu)}\partial X^\mu + \frac{\partial\mathcal{L}}{\partial(\bar{\partial}\psi^\mu)}\partial\psi^\mu + \frac{\partial\mathcal{L}}{\partial(\bar{\partial}^z\tilde{\psi}^\mu)}\partial\tilde{\psi}^\mu \\ &\Rightarrow T^{\bar{z}}{}_z = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \end{aligned} \tag{10.1.11}$$

$$-\frac{1}{2\pi}T^z{}_{\bar{z}} = \frac{\partial\mathcal{L}}{\partial(\partial_z X^\mu)}\partial_{\bar{z}}X^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_z\tilde{\psi}^\mu)}\partial_{\bar{z}}\tilde{\psi}^\mu - g_{\bar{z}\bar{z}}\mathcal{L} = \frac{\partial\mathcal{L}}{2\partial(\partial X^\mu)}\bar{\partial}X^\mu + \frac{\partial\mathcal{L}}{2\partial(\partial\tilde{\psi}^\mu)}\bar{\partial}\tilde{\psi}^\mu$$

$$\Rightarrow T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'}\bar{\partial}X^{\mu}\bar{\partial}X_{\mu} - \frac{1}{2}\tilde{\psi}^{\mu}\partial\tilde{\psi}_{\mu} \tag{10.1.12}$$

It can be shown that $T^{z}{}_{z} = T^{\bar{z}}{}_{\bar{z}} = 0$. We define the stress tensor as follows;

$$T_B(z) = -\frac{1}{\alpha'}\partial X^{\mu}\partial X_{\mu} - \frac{1}{2}\psi^{\mu}\partial\psi_{\mu},\ \tilde{T}_B(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial}X^{\mu}\bar{\partial}X_{\mu} - \frac{1}{2}\tilde{\psi}^{\mu}\partial\tilde{\psi}_{\mu} \tag{10.1.13}$$

The names T_B and $\bar{T}_B(z)$ come from following Polchinski. We will use these names from now on. Now, we define the (1,1) superconformal transformations as follows;

$$\delta X^{\mu} = \sqrt{\frac{\alpha'}{2}}\left(\epsilon(z)\psi^{\mu}(z) + \epsilon^{*}(z)\tilde{\psi}^{\mu}(z)\right)$$

$$\delta\psi^{\mu} = -\sqrt{\frac{2}{\alpha'}}\epsilon(z)\partial X^{\mu}(z)$$

$$\delta\tilde{\psi}^{\mu} = -\sqrt{\frac{2}{\alpha'}}\epsilon^{*}(\bar{z})\bar{\partial}X^{\mu}(\bar{z}) \tag{10.1.14}$$

It can be checked that the action (10.1.5) is invariant under these transformations. We can calculate the quantity that is conserved in these symmetries by using the Noetherian method.

$$\delta S = \frac{1}{2\pi}\int d^2z\left[\partial\epsilon^{*}\left(\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^{\mu}\bar{\partial}X_{\mu}\right) + \bar{\partial}\epsilon\left(\sqrt{\frac{2}{\alpha'}}\psi^{\mu}\partial X_{\mu}\right)\right]$$

and we define the conserved quantities, known as worldsheet supercurrents (following Polchinski);

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^{\mu}\partial X_{\mu}(z),\ \tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^{\mu}\bar{\partial}X_{\mu}(\bar{z}) \tag{10.1.15}$$

Using (10.1.7), we can check that T_B and T_F close an operator algebra (same is the case for anti-holomorphic counterparts). We get;

$$T_B(z)T_B(0) \sim \frac{(3D/2)}{2z^4} + \frac{2T_B(0)}{z^2} + \frac{\partial T_B(0)}{z}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{\partial T_B(0)}{z}$$

$$T_F(z)T_F(0) \sim \frac{D}{z^3} + \frac{2T_B(0)}{z} \tag{10.1.16}$$

which gives us a central charge of $c = 3D/2$ for the theory. It also shows that T_F is a primary field of conformal weight $\left(\frac{3}{2}, 0\right)$. Written in terms of the central charge, we have;

$$T_B(z)T_B(0) \sim \frac{c}{2z^4} + \frac{2T_B(0)}{z^2} + \frac{\partial T_B(0)}{z}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{\partial T_B(0)}{z}$$

$$T_F(z)T_F(0) \sim \frac{2c}{3z^3} + \frac{2T_B(0)}{z} \tag{10.1.17}$$

This operator algebra is called the $(1,1)$ superconformal algebra. He then gives two examples of SCFTs. One of them is the superconformal generalization of the bc ghost system given as follows;

$$S_{bc} = \frac{1}{2\pi}\int d^2z b\bar{\partial}c,\ \ h_b = \lambda,\ \ h_c = 1-\lambda \tag{10.1.18}$$

with the generalization including the fermionic, commuting ghosts named β and γ. The SCFT action is then given as follows;

$$S_{bc\beta\gamma} = \frac{1}{2\pi}\int d^2z\left(b\bar{\partial}c + \beta\bar{\partial}\gamma\right),\ \ h_{\beta} = \lambda - \frac{1}{2},\ h_{\gamma} = \frac{3}{2} - \lambda \tag{10.1.19}$$

10.2 NS and R sectors

If we start with the theory on the cylinder for the closed string (with coordinates $w = \sigma + i\tau, \bar{w} = \sigma - i\tau \Rightarrow z = e^{-iw}, \bar{z} = e^{i\bar{w}}$ with $0 \le \sigma \le 2\pi, -\infty \le \tau \le \infty$), then we have the following cases;

$$\psi^\mu(w+2\pi) = \begin{cases} \psi^\mu(w) \text{ (Ramond)} \\ -\psi^\mu(w) \text{ (Neveu Schwarz)} \end{cases} \Rightarrow \psi^\mu(w+2\pi) = e^{2\pi i\nu}\psi^\mu(w) \text{ where } \begin{cases} \nu = 0 \text{ (Ramond)} \\ \nu = \frac{1}{2} \text{ (Neveu Schwarz)} \end{cases} \tag{10.2.1}$$

Similarly,

$$\tilde{\psi}^\mu(w+2\pi) = \begin{cases} \tilde{\psi}^\mu(w) \text{ (Ramond)} \\ -\tilde{\psi}^\mu(w) \text{ (Neveu Schwarz)} \end{cases} \Rightarrow \tilde{\psi}^\mu(w+2\pi) = e^{2\pi i\tilde{\nu}}\tilde{\psi}^\mu(w) \text{ where } \begin{cases} \tilde{\nu} = 0 \text{ (Ramond)} \\ \tilde{\nu} = \frac{1}{2} \text{ (Neveu Schwarz)} \end{cases} \tag{10.2.2}$$

We will take $X^\mu(w)$ to be periodic. Note that in this choice, $T_B(w)$ is periodic in any of the choices in (10.2.1) but $T_F(w)$ has the same periodicity as $\psi^\mu(w)$ (the same thing goes for the anti-holomorphic sector). For the open string, we take $0 \le \sigma \le \pi$. The holomorphic fermion field on the cylinder can be expanded as follows;

$$\psi^\mu(w) = i^{-1/2} \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_{(c)r} e^{irw} \tag{10.2.3}$$

where the (c) subscript indicates that these are cylinder modes. Now, since the free fermion is a primary field with conformal weight 1/2, we can find the fermion field on the plane as follows;

$$\psi^\mu(z) = \left(\frac{\partial z}{\partial w}\right)^{-1/2} \psi^\mu(w) = \frac{(-i)^{-1/2}}{z^{1/2}}\psi^\mu(w) = \frac{(-i)^{-1/2}}{z^{1/2}} \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_{(c)r} e^{irw} = \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_{(p)r} z^{-r-\frac{1}{2}} \tag{10.2.4}$$

where we defined $\psi^\mu_{(p)r} = (-i)^{-1/2}\psi^\mu_{(c)r}$ as the modes on the plane. We will drop the (p) subscript from now on. **Please note that on the plane, Ramond fermions are anti-periodic and vice versa (which is the opposite of the cylinder story).** Similar expression is obtained for the anti-holomorphic fermionic field;

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \tilde{\psi}^\mu_r \bar{z}^{-r-\frac{1}{2}} \tag{10.2.5}$$

The modes expansions of the bosonic currents are already known to us i.e.

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{\alpha^\mu_m}{z^{m+1}},\ \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}} \tag{10.2.6}$$

Using the following identities;

$$\begin{aligned} \oint dz[\partial X^\mu(z), \partial X^\nu(w)] &= \oint_{C(w)} dz\mathcal{R}[\partial X^\mu(z)\partial X^\nu(w)] \\ \oint dz\{\psi^\mu(z), \psi^\nu(w)\} &= \oint_{C(w)} dz\mathcal{R}[\psi^\mu(z)\psi^\nu(w)] \end{aligned} \tag{10.2.7}$$

where $\mathcal{R}$ means radial ordering (which gives the OPE of the operators of the in this operator), and the OPE of ∂X^μ with itself and OPE of ψ^μ with itself gives us the following (anti) commutators;

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n},\ \{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s} \tag{10.2.8}$$

Using the mode expansions of T_B and T_F

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}},\ T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+\frac{3}{2}}} \tag{10.2.9}$$

we can derive the following algebra;

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r} \end{aligned} \tag{10.2.10}$$

called the RNS algebra or the Ramond algebra for r, s integers and Neveu-Schwarz algebra for r, s half integers.

10.2.1 NS and R spectrum

The NS spectrum is easy to construct. $|0\rangle_{\text{NS}}$ is the ground state (and it is unique) such that;

$$\psi^{\mu}_{r}|0\rangle_{\text{NS}} = 0 \text{ if } r > 0 \tag{10.2.11}$$

The R sector has more than one ground state. If $|0\rangle_{\text{R}}$ is a ground state, then so is $\psi^{\mu}_{0}|0\rangle_{\text{R}}$ and due to (10.1.8), these states form a representation of the Clifford algebra. Seeing the similarity in (21.1.1) and (10.2.8), we identity the following;

$$\Gamma^{\mu} \sim \sqrt{2}\psi^{\mu}_{0}$$

Therefore $e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$ will take Ramond ground states to Ramond ground states and hence, $|0\rangle_{R}$ can be thought of having a spinor index and as living in a representation of Clifford algebra. Since a basis for spinors is given in (21.1.6) and a member of this basis is represented by the k-tuple $\mathbf{s}$, we can choose this basis for the Ramond ground states. So, the basis for R-ground states is as follows;

$$|s_0, ..., s_4\rangle = |\mathbf{s}\rangle \quad s_a = \pm\frac{1}{2} \tag{10.2.12}$$

Because of (21.1.9), we see that half integral values of s_a mean that R-ground states are spinors. We now construct an operator called the **fermion number operator** denoted as F. Since Γ^{μ} is identified as the zero mode of ψ^{μ}, and since Γ anti commutes with Γ^{μ}, we can work out a quantity that anticommutes with the full ψ^{μ}. This quantity is $e^{\pi i F}$ where F is the fermion number operator (which is defined only mod 2) with the following desired property;

$$\begin{aligned} F|\psi\rangle = f|\psi\rangle \Rightarrow F((\psi^{2a}_{s} \pm i\psi^{2a+1}_{s})|\psi\rangle) = (f \pm 1)((\psi^{2a}_{s} \pm i\psi^{2a+1}_{s})|\psi\rangle) = (\psi^{2a}_{s} \pm i\psi^{2a+1}_{s})F|\psi\rangle \pm (\psi^{2a}_{s} \pm i\psi^{2a+1}_{s})|\psi\rangle \\ \Rightarrow [F, (\psi^{2a}_{s} \pm i\psi^{2a+1}_{s})] = \pm(\psi^{2a}_{s} \pm i\psi^{2a+1}_{s}) \end{aligned} \tag{10.2.13}$$

where $|\psi\rangle$ is an arbitrary state and f is just a number. So, ψ^{μ} increases F by one. Using (10.2.13), we get;

$$e^{\pi iF}|\psi\rangle = e^{\pi if}|\psi\rangle \Rightarrow e^{\pi iF}\psi^{\mu}|\psi\rangle = e^{\pi i(f+1)}\psi^{\mu}|\psi\rangle = -\psi^{\mu}e^{\pi if}|\psi\rangle = -\psi^{\mu}e^{\pi iF}|\psi\rangle \Rightarrow \{e^{\pi iF}, \psi^{\mu}\} = 0$$

So, if we can construct an F that satisfies (10.2.13), then $e^{\pi iF}$ satisfies our requirements. Before constructing F, we extend $\Sigma^{\alpha\beta}$ and S_a as well (like we extended Γ^{μ} and Γ). Here are the extensions;

$$\begin{aligned} \Sigma^{\alpha\beta} &= -\frac{i}{4}[\Gamma^{\alpha}, \Gamma^{\beta}] = -\frac{i}{2}[\psi^{\alpha}_{0}, \psi^{\beta}_{0}] \to -\frac{i}{2}\sum_{r\in\mathbb{Z}+\nu}[\psi^{\alpha}_{r}, \psi^{\beta}_{-r}] = -i\sum_{r\in\mathbb{Z}+\nu}\psi^{\alpha}_{r}\psi^{\beta}_{-r} \\ S_a &= i^{\delta_{a,0}}\Sigma^{2a,2a+1} \Rightarrow S_0 = \sum_{r\in\mathbb{Z}+\nu}\psi^{0}_{r}\psi^{1}_{-r}, \quad S_a = -i\sum_{r\in\mathbb{Z}+\nu}\psi^{2a}_{r}\psi^{2a+1}_{-r}, \quad (a \neq 0) \end{aligned} \tag{10.2.14}$$

Now, the claim is that F is given as follows;

$$F = S_0 + S_1 + S_2 + S_3 + S_4 \tag{10.2.15}$$

We now derive some anti-commutation relations. For $a = 1, 2, 3, 4$, we have the following two anti-commutation relations;

$$\begin{aligned} S_a\psi^{2a}_{s} &= -i\sum_{r\in\mathbb{Z}+\nu}\psi^{2a}_{r}\psi^{2a+1}_{-r}\psi^{2a}_{s} = i\psi^{2a+1}_{s} - i\psi^{2a}_{s}\sum_{r\in\mathbb{Z}+\nu}\psi^{2a}_{r}\psi^{2a+1}_{r} \\ &= i\psi^{2a+1}_{s} + \psi^{2a}_{s}S_a \Rightarrow [S_a, \psi^{2a}_{s}] = [F, \psi^{2a}_{s}] = i\psi^{2a+1}_{s} \\ S_a\psi^{2a+1}_{s} &= -i\sum_{r\in\mathbb{Z}+\nu}\psi^{2a}_{r}\psi^{2a+1}_{-r}\psi^{2a+1}_{s} = -i\psi^{2a}_{s} - i\psi^{2a+1}_{s}\sum_{r\in\mathbb{Z}+\nu}\psi^{2a}_{r}\psi^{2a+1}_{r} \\ &= -i\psi^{2a}_{s} + \psi^{2a+1}_{s}S_a \Rightarrow [S_a, \psi^{2a+1}_{s}] = [F, \psi^{2a+1}_{s}] = -i\psi^{2a}_{s} \end{aligned}$$

The above two relations give the following;

$$[F, \psi^{2a}_{s} \pm i\psi^{2a+1}_{s}] = i\psi^{2a+1}_{s} \pm i(-i\psi^{2a}_{s}) = \pm(\psi^{2a}_{s} \pm i\psi^{2a+1}_{s}) \tag{10.2.16}$$

which is consistent with (10.2.13). Moreover, for the remaining spinors, we have the following;

$$S_0\psi^{0}_{s} = \sum_{r\in\mathbb{Z}+\nu}\psi^{0}_{r}\psi^{1}_{-r}\psi^{0}_{s} = -\sum_{r\in\mathbb{Z}+\nu}\psi^{0}_{r}\psi^{0}_{s}\psi^{1}_{-r} = -\sum_{r\in\mathbb{Z}+\nu}(-\delta_{r+s} - \psi^{0}_{s}\psi^{0}_{r})\psi^{1}_{-r}$$

$$= \psi_s^1 + \psi_s^0 \sum_{r\in\mathbb{Z}+\nu} \psi_r^0 \psi_{-r}^1 = \psi_s^1 + \psi_s^0 S_0 \Rightarrow [S_0, \psi_s^0] = \psi_s^1$$

$$S_0 \psi_s^1 = \sum_{r\in\mathbb{Z}+\nu} \psi_r^0 \psi_{-r}^1 \psi_s^1 = \sum_{r\in\mathbb{Z}+\nu} \psi_r^0 (\delta_{-r+s} - \psi_s^1 \psi_{-r}^1)$$

$$= \psi_s^0 + \psi_s^1 \sum_{r\in\mathbb{Z}+\nu} \psi_r^0 \psi_{-r}^1 = \psi_s^0 + \psi_s^1 S_0 \Rightarrow [S_0, \psi_s^1] = \psi_s^0$$

Therefore, we get;

$$[F, \psi_r^0 \pm i\psi_r^1] = \psi_r^1 \pm i\psi_r^0 = \pm i(\psi_r^0 \mp i\psi_r^1)$$

This doesn't agree with (10.2.13)**. We will get back to that**. An alternative definition of F (e.g. in Blumenhagen) is as follows (in NS sector);

$$F = \sum_{r>0} \sum_{a=2}^{8} \psi_{-r}^a \psi_r^a - 1 \tag{10.2.17}$$

This implies the following (for $b > 2$);

$$[F, \psi_s^b] = \sum_{r>0} \sum_{a=2}^{8} [\psi_{-r}^a \psi_r^a, \psi_s^b] = \sum_{r>0} \sum_{a=2}^{8} (\psi_{-r}^a \psi_r^a \psi_s^b - \psi_s^b \psi_{-r}^a \psi_r^a) = \sum_{r>0} \sum_{a=2}^{8} \eta^{ab} (\delta_{r+s} \psi_{-r}^a - \delta_{s-r} \psi_r^a)$$

$$= \sum_{r>0} (\delta_{r+s} \psi_{-r}^b - \delta_{s-r} \psi_r^b)$$

Now, we have the following two cases;

$$[F, \psi_s^b] = \sum_{r>0} (\delta_{r+s} \psi_{-r}^b - \delta_{s-r} \psi_r^b) = -\psi_s^b \quad (s > 0)$$

$$[F, \psi_s^b] = \sum_{r>0} (\delta_{r+s} \psi_{-r}^b - \delta_{s-r} \psi_r^b) = \psi_s^b \quad (s < 0)$$

which implies the following;

$$[F, \psi_{\pm s}^b] = \mp \psi_{\pm s}^b \quad (s > 0)$$

and this in turn implies the following;

$$F|\psi\rangle = f|\psi\rangle \Rightarrow F(\psi_{\pm s}^b |\psi\rangle) = (\psi_{\pm s}^b F \mp \psi_{\pm s}^b)|\psi\rangle = (f \mp 1)\psi_{\pm s}^b |\psi\rangle$$

So, ψ_s^b decreases F by one and ψ_{-s}^b increases F by one.

10.2.2 Closed string spectra

In the closed string case, the NS-NS vacuum is $|0\rangle_{\text{NS}} \otimes |0\rangle_{\text{NS}}$ and thus, the vacuum has integer spin. In the R-R sector, the vacuum is $|\mathbf{s}, \mathbf{s}'\rangle = |\mathbf{s}\rangle \otimes |\mathbf{s}'\rangle$. This vacuum lies in $\mathbf{32}_{\text{Dirac}} \times \mathbf{32}_{\text{Dirac}}$ representation and using (21.1.35), we can break it down as follows;

$$\mathbf{32}_{\text{Dirac}} \times \mathbf{32}_{\text{Dirac}} = [0]^2 + [1]^2 + [2]^2 + [3]^2 + [4]^2 + [5]$$

If the left and right vaccua have a definite chirality (which is the same as $e^{\pi i F}$ on the vaccua), the vaccua have a definite fermion number on left F and on the the right $\tilde{F}$. For different choices of $F, \tilde{F}$, the breakdown of RR vaccua is as follows (using (21.1.37));

F	$\tilde{F}$	$e^{\pi i F}$	$e^{\pi i \tilde{F}}$	SO(1,9) rep
0	0	1	1	$[1] + [3] + [5]_+$
0	1	1	-1	$[0] + [2] + [4]$
1	0	-1	1	$[0] + [2] + [4]$
1	1	-1	-1	$[1] + [3] + [5]_-$

The NS-R or R-NS vaccua have to be spacetime fermions because they contain one fermion index.

10.3 Vertex operators and bosonization

Using (10.2.4), we see that in the NS sector (with $\nu = 1/2$), the fermion is single-valued. Moreover, (10.1.7), we see that $\psi\psi$ OPE is single value. Hence, the products and derivatives of ψ in the NS sector must be in the NS sector. We now calculate the state corresponding to the $k-$th derivative of ψ in the NS sector.

$$\partial^k \psi^\mu(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi^\mu_r \frac{(-r-\frac{1}{2})!}{(-r-k-\frac{1}{2})!} z^{-r-k-\frac{1}{2}}$$

So, the state corresponding to $\partial^k \psi^\mu$ is as follows;

$$|\partial^k \psi^\mu\rangle = \lim_{z\to 0} \partial^k \psi^\mu(z)|0\rangle = \lim_{z\to 0} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \frac{(-r-\frac{1}{2})!}{(-r-k-\frac{1}{2})!} z^{-r-k-\frac{1}{2}} \psi^\mu_r |0\rangle$$

Now, $\psi^\mu_r|0\rangle$ have to be zero for $r + k > 1/2$ for all k. The tightest bound comes from $k = 0$ and thus;

$$\psi^\mu_r|0\rangle = 0, \;\; r = \frac{1}{2}, \frac{3}{2}, \dots$$

So, we get;

$$|\partial^k \psi^\mu\rangle = k!\, \psi^\mu_{-k-\frac{1}{2}}|0\rangle$$

So, we see that for every derivative, we have one state in the NS sector.
For the Ramond vertex operators, we need bosonization. Let $H(z)$ be a holomorphic field (with $\alpha' = 2$). The OPE's can be calculated using the following;

$$\mathcal{V}_{k_L,k_R}(z_1,\bar{z}_1)\mathcal{V}_{k'_L,k'_R}(z_2,\bar{z}_2) \sim z_{12}^{\alpha' k_L k'_L/2} \bar{z}_{12}^{\alpha' k_R k'_R/2} \mathcal{V}_{k_L+k'_L,k_R+k_R}(z_2,\bar{z}_2)$$
$$\text{where } \mathcal{V}_{k_L,k_R}(z,\bar{z}) = e^{ik_L X_L(z) + ik_R X_R(\bar{z})} \tag{10.3.1}$$

The OPE's are as follows;

$$e^{iH(z)}e^{-iH(z)} \sim \frac{1}{z}, \;\; e^{iH(z)}e^{iH(z)} \sim \mathcal{O}(z) \;\; e^{-iH(z)}e^{-iH(z)} \sim \mathcal{O}(z) \tag{10.3.2}$$

Now, consider two Majorana-Weyl fermions;

$$\psi(z) = \frac{1}{\sqrt{2}}(\psi^1(z) + i\psi^2(z)), \;\; \bar{\psi}(z) = \frac{1}{\sqrt{2}}(\psi^1(z) - i\psi^2(z)) \tag{10.3.3}$$

and the OPE's for fermions is calculated using (10.1.7) as follows;

$$\psi(z)\bar{\psi}(z) \sim \frac{1}{2}(\psi^1(z)\psi^1(z) + \psi^2(z)\psi^2(z)) = \frac{1}{z}, \;\; \psi(z)\psi(z) \sim (O)(z), \;\; \bar{\psi}(z)\bar{\psi}(z) \sim (O)(z) \tag{10.3.4}$$

We identify $\psi \sim e^{iH(z)}$ and $\bar{\psi}(z) \sim e^{-iH(z)}$. This identification goes to all the operators as they can be built using these fundamental operators. Using the following OPEs (**derive using Taylor's expansion**);

$$e^{iH(z)}e^{-iH(z)} \sim \frac{1}{2z} + i\partial H(0) + 2zT^H_B(0) + \mathcal{O}(z^2)$$

$$\psi(z)\psi(-z) \sim \frac{1}{2z} + \psi\bar{\psi}(0) + 2zT^\psi_B(0) + \mathcal{O}(z^2)$$

So, the stress tensors match and we have $\psi\bar{\psi} \sim i\partial H$. We can see (**derive this**) using BCH formula that $e^{iH(z)}$ anti-commute with each other. Now, if we consider the general periodicity condition for (10.3.3) (as compared to (10.2.1));

$$\psi(w + 2\pi) = e^{2\pi i\nu}\psi(w), \;\; \nu \in \mathbb{R} \tag{10.3.5}$$

and we define a reference state (**is it a general state**) as follows (it is not the vacuum);

$$\psi_{n+\nu}|0\rangle_\nu = \bar{\psi}_{n-\nu+1}|0\rangle_\nu = 0 \tag{10.3.6}$$

This gives the Laurent series of $\psi(z)$ and $\bar{\psi}(z)$ as follows;

$$\psi(z) = \sum_{n=1}^{\infty} \frac{\psi_{\nu-n}}{z^{\nu-n+1/2}} \sim \mathcal{O}(z^{1/2-\nu}) \Rightarrow \psi(z)\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{1/2-\nu})$$

$$\bar{\psi}(z) = \sum_{n=0}^{\infty} \frac{\psi_{-\nu-n}}{z^{-\nu-n+1/2}} \sim \mathcal{O}(z^{\nu-1/2}) \Rightarrow \bar{\psi}(z)\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{\nu-1/2})$$

where $\mathcal{A}_\nu$ is the corresponding operator for $|0\rangle_\nu$. Using the identifications for $\psi(z)$ and $\bar{\psi}(z)$ from before, we deduce the following;

$$\psi(z)\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{1/2-\nu}) \Rightarrow e^{iH(z)}\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{1/2-\nu}) \sim e^{iH(z)}e^{i(1/2-\nu)H(z)} \Rightarrow \mathcal{A}_\nu(z) \sim e^{i(1/2-\nu)H(z)}$$

$$\bar{\psi}(z)\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{\nu-1/2}) \Rightarrow e^{-iH(z)}\mathcal{A}_\nu(0) \sim \mathcal{O}(z^{\nu-1/2}) \sim e^{-iH(z)}e^{i(1/2-\nu)H(z)} \Rightarrow \mathcal{A}_\nu(z) \sim e^{i(1/2-\nu)H(z)}$$

So, we get a consistent expression for $\mathcal{A}_\nu$. The weight of $\mathcal{A}_\nu$ is easily seen to be as follows;

$$h(\mathcal{A}_\nu) = \frac{1}{2}\left(\nu - \frac{1}{2}\right)^2$$

The fermionic way to derive this weight. We see that (10.3.5) is the same if $\nu \to \nu + 1$ but the state in (10.3.6) changes. It the ground state for $0 \le \nu \le 1$ (**WHY?**). Now, we will show that $|0\rangle_{\nu+1} = \bar{\psi}_{-\nu}|0\rangle_\nu$ by showing that this satisfies (10.3.6) with $\nu \to \nu + 1$.

$$\psi_{n+\nu+1}|0\rangle_{\nu+1} = \psi_{n+\nu+1}\bar{\psi}_{-\nu}|0\rangle_\nu = -\bar{\psi}_{-\nu}\psi_{n+\nu+1}|0\rangle_\nu = 0$$

$$\bar{\psi}_{n-\nu}|0\rangle_{\nu+1} = \bar{\psi}_{n-\nu}\bar{\psi}_{-\nu}|0\rangle_\nu = -\bar{\psi}_{-\nu}\bar{\psi}_{n-\nu}|0\rangle_\nu = \begin{cases} 0 & \text{if } n \neq 0 \ \ (|0\rangle_\nu\text{condition}) \\ 0 & \text{if } n = 0 \ \ (\bar{\psi}_{-\nu}\bar{\psi}_{-\nu} = 0) \end{cases}$$

This phenomenon is called **spectral flow**. For the Ramond case, we have $\nu = 0$ but the state $|0\rangle_0$ will flow into $|0\rangle_1$. The corresponding operators for these states are as follows;

$$|0\rangle_0 \sim e^{iH(z)/2}, \ \ |0\rangle_1 \sim e^{-iH(z)/2}$$

We relabel these states as follows;

$$|s\rangle \sim e^{isH(z)}, \ \ s = \frac{1}{2}$$

This resembles the ground state of the Ramond sector. However, for Ramond sector, we need five different s values and thus, five different H's. These are dual to five different complex fermions. They are constructed as follows;

$$\frac{1}{\sqrt{2}}(\pm\psi^0(z) + \psi^1(z)) \sim e^{\pm iH^0}$$

$$\frac{1}{\sqrt{2}}(\pm\psi^{2a}(z) + \psi^{2a+1}(z)) \sim e^{\pm iH^a}, \ \ a = 1, 2, 3, 4$$

Thus finally, we can come up with the vertex operator for the Ramond ground state (denoted as $\Theta_{\mathbf{s}}$) as follows;

$$|\mathbf{s}\rangle \sim \Theta_{\mathbf{s}} = \exp\left[i\sum_a s_a H^a\right]$$

$\Theta_{\mathbf{s}}$ is also called a **spin field**. **(Bosonization of the bc ghost system and equivalence to linear dilaton)**

10.4 The superconformal ghosts

The action of the superconfromal ghost action is as follows;

$$S = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma)$$

and the stress tensor for this theory is as follows (see (2.5.5));

$$T_B(z) =: \partial bc : (z) - \lambda\partial : bc : (z) + : \partial\beta\gamma : (z) - \left(\lambda - \frac{1}{2}\right)\partial : \beta\gamma : (z) \tag{10.4.1}$$

If we choose $\lambda = 2$, we get;

$$T_B(z) =: \partial bc : (z) - 2\partial : bc : (z) + : \partial\beta\gamma : (z) - \frac{3}{2}\partial : \beta\gamma : (z) \tag{10.4.2}$$

The expression for T_F is (**derive this**);

$$T_F(z) =: \partial\beta c : (z) + \frac{3}{2} : \beta\partial c : (z) - 2 : b\gamma : (z) \tag{10.4.3}$$

So, the conformal weights of the fields are as follows;

$$h(b) = 2,\ h(c) = -1;\ h(\beta) = \frac{3}{2};\ h(\gamma) = -\frac{1}{2} \tag{10.4.4}$$

and the mode expansions are thus as follows (using the fact that the periodicities of these ghosts should be the same as T_F);

$$b(z) = \sum_{m\in\mathbb{Z}}\frac{b_m}{z^{m+2}},\ c(z) = \sum_{m\in\mathbb{Z}}\frac{c_m}{z^{m-1}},\ \beta(z) = \sum_{r\in\mathbb{Z}+\nu}\frac{\beta_r}{z^{r+3/2}},\ \gamma(z) = \sum_{n\in\mathbb{Z}+\nu}\frac{\gamma_r}{z^{r-1/2}}, \tag{10.4.5}$$

Now we define the ground state as follows;

$$\begin{gathered}\beta_r|0\rangle_{\text{NS}} = 0,\ r \geq \frac{1}{2},\ \ \gamma_r|0\rangle_{\text{NS}} = 0,\ r \geq \frac{1}{2}\\ \beta_r|0\rangle_{\text{R}} = 0,\ r \geq 0,\ \ \gamma_r|0\rangle_{\text{R}} = 0,\ r \geq 1\\ b_m|0\rangle_{\text{NS,R}} = 0,\ m \geq 0,\ \ c_m|0\rangle_{\text{NS,R}} = 0,\ m \geq 1\end{gathered} \tag{10.4.6}$$

Notice that in the NS case, β_r with positive r's are annihilation operators and thus, their conjugates (based on the commutation relations) γ_{-r}'s are creation operators. The same thing happens with the R case but they problem of zero modes is there. We choose β_0 to be an annihilation operator and thus, its conjugate γ_0 is the creation operator. A similar story happens but b_m's and c_m's (b_0 is chosen to be the annihilation operator).

10.4.1 Vertex operators

We consider the state corresponding to the unit operator. Let's call it $|1\rangle$. It has to be in the NS sector because the unit operator doesn't have a branch. Now, this state has to satisfy the following properties (see (10.4.4));

$$\beta_r|1\rangle = 0,\ \left(r > -\frac{3}{2}\right),\ \gamma_r|1\rangle = 0,\ \left(r > \frac{1}{2}\right),\ b_m|1\rangle = 0,\ (m > -2),\ c_m|1\rangle = 0,\ (r > 1) \tag{10.4.7}$$

$|1\rangle$ is not $|0\rangle_{\text{NS}}$. We can show that $|0\rangle_{\text{NS}} = c_1|1\rangle$. It trivially satisfies all the properties of $|0\rangle_{\text{NS}}$ as listed in (10.4.6). We now want to find the vertex operator for $|0\rangle_{\text{NS}}$ in terms of γ. Let's call it $\delta(\gamma(z))$ i.e.

$$|0\rangle_{\text{NS}} = \delta(\gamma(0))|1\rangle$$

. We now derive the following results;

$$\gamma(z)\delta(\gamma(0))|1\rangle = \sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\text{NS}} = \sum_{r<\frac{1}{2}}\frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\text{NS}} = \gamma_{-\frac{1}{2}}z|0\rangle_{\text{NS}} + ...$$

$$\sim \mathcal{O}(z)|0\rangle_{\rm NS} \Rightarrow \gamma(z)\delta(\gamma(0)) \sim \mathcal{O}(z) \tag{10.4.8}$$

$$\beta(z)\delta(\gamma(0))|1\rangle = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\rm NS} = \sum_{r<\frac{1}{2}} \frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\rm NS} = \frac{1}{z}\beta_{-\frac{1}{2}}|0\rangle_{\rm NS} + \ldots$$

$$\sim \mathcal{O}(z^{-1})|0\rangle_{\rm NS} \Rightarrow \beta(z)\delta(\gamma(0)) \sim \mathcal{O}(z^{-1}) \tag{10.4.9}$$

Now we will bosonize the $\beta\gamma$ system to look it like the bosonisation of bc system. We consider the following OPE;

$$\beta(z)\gamma(z)\beta(0)\gamma(0) = \beta(z)\gamma(0)\gamma(z)\beta(0) \sim -\frac{1}{z^2} \tag{10.4.10}$$

This matches the OPE of $\partial\phi$ with itself where ϕ is the chiral, free boson. So, we have the following correspondence;

$$\beta\gamma(z) \sim \partial\phi(z) \tag{10.4.11}$$

Moreover, we can get another correspondence. Firstly, we get the following results;

$$\beta(z)\gamma(z)\beta(0) \sim \frac{\beta(0)}{z} \Rightarrow \partial\phi(z)\beta(0) \sim \frac{\beta(0)}{z},\ \beta(z)\gamma(z)\gamma(0) \sim -\frac{\gamma(0)}{z} \Rightarrow \partial\phi(z)\gamma(0) \sim -\frac{\gamma(0)}{z}$$

and then, we consider the following OPEs

$$\partial\phi(z)e^{-\phi(0)} \sim \frac{e^{-\phi(0)}}{z},\ \partial\phi(z)e^{\phi(0)} \sim -\frac{e^{\phi(0)}}{z}$$

Comparing the OPEs in the above two equations, we get the following;

$$\beta(z) \sim e^{-\phi(z)},\ \gamma(z) \sim e^{\phi(z)} \tag{10.4.12}$$

This will cause a problem. Consider the following OPEs;

$$e^{\pm\phi(z)}e^{\pm\phi(0)} \sim \mathcal{O}(z^{-1}),\ e^{\phi(z)}e^{-\phi(0)} \sim \mathcal{O}(z) \tag{10.4.13}$$

this implies that we will have the following OPEs;

$$\beta(z)\beta(0) \sim \mathcal{O}(z^{-1}),\ \gamma(z)\gamma(0) \sim \mathcal{O}(z^{-1}),\ \beta(z)\gamma(0) \sim \mathcal{O}(z)$$

but these are the wrong OPEs. We can fix this by adding additional fields which are non-singular with respect to ϕ in order to retain the OPE in (10.4.10). The new bosonization is as follows;

$$\beta(z) \sim e^{-\phi(z)}\partial\xi(z),\ \gamma(z) \sim e^{\phi(z)}\eta(z) \tag{10.4.14}$$

and we need to impose the $\beta\gamma$ OPEs to get the OPEs between $\partial\xi$ and η as follows;

$$\beta(z)\gamma(0) \sim e^{-\phi(z)}\partial\xi(z)e^{\phi(0)}\eta(0) = e^{-\phi(z)}e^{\phi(0)}\partial\xi(z)\eta(0) \sim \mathcal{O}(z^{-1}) \Rightarrow \xi(z)\eta(0) \sim \mathcal{O}(z^{-1})$$

$$\beta(z)\beta(0) \sim e^{-\phi(z)}\partial\xi(z)e^{-\phi(0)}\partial\xi(0) = e^{-\phi(z)}e^{-\phi(0)}\partial\xi(z)\partial\xi(0) \sim \mathcal{O}(1) \Rightarrow \partial\xi(z)\partial\xi(0) \sim \mathcal{O}(z)$$

$$\gamma(z)\gamma(0) \sim e^{\phi(z)}\eta(z)e^{\phi(0)}\eta(0) = e^{\phi(z)}e^{\phi(0)}\eta(z)\eta(0) \sim \mathcal{O}(1) \Rightarrow \eta(z)\eta(0) \sim \mathcal{O}(z)$$

This makes this theory look like a bc theory (although this will imply that $\partial\xi(z)\partial\xi(0) \sim \mathcal{O}(1)$ but the $\mathcal{O}(1)$ term should vanish due to the anti commutativity). Now, we determine the stress tensor of the ϕ theory. The stress tensor of $\beta\gamma$ theory (with β having a weight of λ) is as follows (I have excluded the normal order sign);

$$T(z) = (1-\lambda)\partial\beta(z)\gamma(z) - \lambda\beta(z)\partial\gamma(z)$$

The OPE of this $T(z)$ can be calculated with $\beta\gamma$ but we will calculate the fully contracted term only. The calculation is as follows;

$$T(z)\beta(0)\gamma(0) \sim (1-\lambda)\partial\beta(z)\gamma(z)\beta(0)\gamma(0) - \lambda\beta(z)\partial\gamma(z)\beta(0)\gamma(0) + \ldots = \frac{(1-\lambda)}{z^3} - \frac{\lambda}{z^3} + \ldots = \frac{1-2\lambda}{z^3} + \ldots$$

This can be compared with the following OPE;

$$-\frac{1}{2}\partial\phi(z)\partial\phi(z)\partial\phi(0)-\frac{1}{2}\partial\phi(z)\partial\phi(z)\partial\phi(0)+\frac{1}{2}(1-2\lambda)\partial^2\phi(z)\partial\phi(0)=\frac{1-2\lambda}{z^3}+\frac{\partial\phi(0)}{z^2}+\frac{\partial^2\phi(0)}{z}$$

This gives the conclusion that the stress tensor for the ϕ theory is as follows;

$$T_B^{\phi}(z)=-\frac{1}{2}\partial\phi(z)\partial\phi(z)+\frac{1-2\lambda}{2}\partial^2\phi(z) \tag{10.4.15}$$

The weights of the exponentials in (10.4.14) are calculated as follows;

$$T_B^{\phi}(z)e^{\phi(0)}\sim-\frac{1}{2}\partial\phi(z)\partial\phi(z)e^{\phi(0)}+\frac{1-2\lambda}{2}\partial^2\phi(z)e^{\phi(0)}\sim\frac{1/2-\lambda}{z^2}e^{\phi(0)}+\frac{\partial\phi(z)e^{\phi(0)}}{2z}\sim-\frac{\lambda}{z^2}e^{\phi(0)}+\mathcal{O}(z^{-1})$$

$$T_B^{\phi}(z)e^{-\phi(0)}\sim-\frac{1}{2}\partial\phi(z)\partial\phi(z)e^{-\phi(0)}+\frac{1-2\lambda}{2}\partial^2\phi(z)e^{-\phi(0)}\sim\frac{\lambda-1/2}{z^2}e^{-\phi(0)}-\frac{\partial\phi(z)e^{-\phi(0)}}{2z}\sim\frac{\lambda-1}{z^2}e^{\phi(0)}+\mathcal{O}(z^{-1})$$

which means that the weights of $\xi(z)$ and $\eta(z)$ have to be 0 and 1 respectively. Therefore, the $\eta\xi$ system is a $\lambda=1$ bc system with η being like b and ξ being like c. Using (2.5.5), the stress tensor of $\eta\xi$ theory is as follows;

$$T_B^{\eta\xi}=-:\eta\partial\xi:(z) \tag{10.4.16}$$

We now calculate the central charges of the ϕ theory and the $\eta\xi$ theory. Observing (10.4.15), we see that it is a one-dimensional linear dilaton CFT with $\alpha'=2$ and $V=(1-2\lambda)/2$. So, the central charge of ϕ theory is as follows;

$$c_\phi=1+12V^2=1+12\left(\frac{1-2\lambda}{2}\right)^2=3(1-2\lambda)^2+1 \tag{10.4.17}$$

Moreover, since $\eta\gamma$ is equivalent to a bc system with $\lambda=1$ we can use (2.5.6) to get the following;

$$c_{\eta\gamma}=1-3(2(1)-1)^2=-2 \tag{10.4.18}$$

So, the central charge of the $\beta\gamma$ system is $3(1-2\lambda)^2+1-2=3(1-2\lambda)^2-1$ which is the right central charge. **Write about the representation of beta gamma system in terms of the free boson**.
For string theory, the relevant value of λ is $3/2$. Using (10.4.9) and (10.4.8), we can identify $\delta(\gamma(z))$ as the following;

$$\delta(\gamma(z))\sim e^{-\phi(z)} \tag{10.4.19}$$

which has $h=1/2$. Now, we can do a similar analysis for the vertex operator of R ground state. Let's call it Σ. It means that we have;

$$|0\rangle_R=\mathcal{V}_{\mathbf{s}}(0|1\rangle=\Sigma(0)\Theta_{\mathbf{s}}(0)|1\rangle \tag{10.4.20}$$

Using (10.4.6) and (10.4.5), we can derive the following;

$$\beta(z)\Sigma(0)|1\rangle=\sum_{r\in\mathbb{Z}}\frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{R}}=\sum_{r<0}\frac{\beta_r}{z^{r+\frac{3}{2}}}|0\rangle_{\mathrm{R}}=\beta_{-1}z^{-1/2}|0\rangle_{\mathrm{R}}+...\sim\mathcal{O}(z^{-1/2})\sim\beta(z)\Sigma(0)\sim\mathcal{O}(z^{-1/2})$$

$$\gamma(z)\Sigma(0)|1\rangle=\sum_{r\in\mathbb{Z}}\frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{R}}=\sum_{r<1}\frac{\gamma_r}{z^{r-\frac{1}{2}}}|0\rangle_{\mathrm{R}}=\gamma_0 z^{1/2}|0\rangle_{\mathrm{R}}+...\sim\mathcal{O}(z^{1/2})\sim\gamma(z)\Sigma(0)\sim\mathcal{O}(z^{1/2})$$

We can compare these OPEs with the following OPEs;

$$e^{\phi(z)}e^{-\phi(0)/2}\sim\mathcal{O}(z^{1/2}),\ e^{-\phi(z)}e^{-\phi(0)/2}\sim\mathcal{O}(z^{-1/2})$$

Using these OPEs and (10.4.14), we can get the following identification;

$$\Sigma(z)\sim e^{-\phi(z)/2} \tag{10.4.21}$$

The worldsheet fermion number F is defined to be odd for β and γ because $\{F,T_F\}=0$ and we want F to commute with BRST charge which contains the term γT_F (**write more about this**). This justifies the negative fermion number of $|0\rangle_{\mathrm{NS}}$. Another way to see F is that we see it as the charge associated to the current in (10.4.11) and this charge is l for $e^{l\phi(z)}$ (**Derive this**). Since the NS vaccum has $e^{-l\phi}$ in its vacuum, its fermion number is -1 (**why R doesn't have $F=-1/2$ then? because of the spin field?**).
(Write about cocycles)

10.5 Physical states

The constraints are applied on the physical states as follows;

$$L_n|\psi\rangle = 0,\ G_r|\psi\rangle = 0,\quad n \geq 0, r \geq 0 \tag{10.5.1}$$

where we only talk about the matter generators right now. The mass-shell condition is as follows;

$$L_0|\psi\rangle = H|\psi\rangle = 0 \tag{10.5.2}$$

where this generator is the matter+ghost generator. H is derived as follows (**derive this**);

$$H = \begin{cases} \alpha' p^2 + N - \frac{1}{2} \text{ (Neveu Schwarz)} \\ \alpha' p^2 + N \text{ (Ramond)} \end{cases} \tag{10.5.3}$$

The NS ground state is $|k,0\rangle_{\text{NS}}$ and is labelled by a momentum k. The constraints (10.5.1) are trivially satisfied for this state. The mass shell condition gives the following;

$$L_0|k,0\rangle_{\text{NS}} = \left(\alpha' p^2 + N - \frac{1}{2}\right)|k,0\rangle_{\text{NS}} = 0 \Rightarrow k^2 = \frac{1}{2\alpha'}$$

a positive k^2 indicates a tachyon. We will call the theory built on this vacuum NS− (because $e^{\pi i F} = -1$). **(Calculate the fermion number by adding ghost contribution)**. The first excited state is given as follows;

$$|e;k\rangle_{\text{NS}} = e_\mu \psi^\mu_{-1/2}|0;k\rangle_{\text{NS}} \tag{10.5.4}$$

where e_μ is the polarization vector. The constraints are calculated as follows;

$$L_m|e;k\rangle_{\text{NS}} = \frac{1}{4}\sum_{r>0}(2r-m)\psi^\mu_{m-r}\psi_{\mu r}e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\text{NS}} + \frac{1}{4}\sum_{r>0}(2r+m)\psi^\mu_r\psi_{\mu m+r}e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\text{NS}}$$

$$= \frac{1}{4}(1-3m)\psi^\mu_{2m-\frac{1}{2}}e_\mu|0;k\rangle_{\text{NS}} + \frac{1}{4}(m+1)\psi^\mu_{\frac{1}{2}}e_\mu|0;k\rangle_{\text{NS}} = 0 \ \ (m>0)$$

So, this constraint is trivially satisfied. The next constraint is imposed as follows;

$$G_r|e;k\rangle_{\text{NS}} = \left(\alpha^\mu_0\psi_{\mu r} + \sum_{n>0}\alpha^\mu_n\psi_{\mu r-n} + \sum_{n>0}\alpha^\mu_{-n}\psi_{\mu r+n}\right)e_\nu\psi^\nu_{-1/2}|0;k\rangle_{\text{NS}}$$

$$= \alpha^\mu_0 e_\mu\delta_{r-\frac{1}{2}}|e;k\rangle_{\text{NS}} + \sum_{n>0}\alpha^\mu_{-n}e_\mu\delta_{r+n-\frac{1}{2}}|e;k\rangle_{\text{NS}} = \alpha^\mu_0 e_\mu\delta_{r-\frac{1}{2}}|e;k\rangle_{\text{NS}}$$

So, the only non-trivial constraint comes from $G_{\frac{1}{2}}$. This constraint implies the following;

$$\alpha^\mu_0 e_\mu|e;k\rangle_{\text{NS}} = 0 \Rightarrow k^\mu e_\mu = 0$$

Finally, the mass shell constraint gives the following;

$$\left(\alpha' p^2 + N - \frac{1}{2}\right)|e;k\rangle_{\text{NS}} = \left(\alpha' k^2 + \frac{1}{2} - \frac{1}{2}\right)|e;k\rangle_{\text{NS}} = 0 \Rightarrow k^2 = 0$$

The fermion number for this state has to be +1. We now show that $G_{-1/2}|e;k\rangle_{\text{NS}}$ is a null state. It is done as follows;

$$\langle\psi|G_{-1/2}|e;k\rangle_{\text{NS}} = \overline{\langle e;k|G_{-1/2}|\psi\rangle} = 0$$

$$L_m G_{-\frac{1}{2}}|e;k\rangle_{\text{NS}} = G_{-\frac{1}{2}}L_m|e;k\rangle_{\text{NS}} + \frac{m+1}{2}G_{m-\frac{1}{2}}|e;k\rangle_{\text{NS}} = 0 \ \ (m>0)$$

$$G_r G_{-\frac{1}{2}}|e;k\rangle_{\text{NS}} = 2L_{r-\frac{1}{2}}|e;k\rangle_{\text{NS}} = 0 \ \ \left(r \geq \frac{1}{2}\right)$$

The form of this null state is as follows;

$$G_{-\frac{1}{2}}|e;k\rangle_{\text{NS}} = \alpha^\mu_0\psi_{\mu-\frac{1}{2}}|e;k\rangle_{\text{NS}} = \sqrt{2\alpha'}k^\mu\psi_{\mu-\frac{1}{2}}|e;k\rangle_{\text{NS}}$$

Thus, the general first excited state is as follows;

$$(e_\mu + \lambda k_\mu)\psi^\mu_{-\frac{1}{2}}|e;k\rangle_{\rm NS} \Rightarrow e_\mu \sim e_\mu + \lambda k_\mu$$

We will refer to the theory built on this vacuum NS+ theory (because $e^{\pi i F} = 1$ on this vacuum). Now, a general Ramond vacuum can be expanded in terms of the basis states labelled by $\mathbf{s}$ as follows;

$$|u;k\rangle_R = |\mathbf{s};k\rangle_R u_{\mathbf{s}} \tag{10.5.5}$$

The mass shell condition calculation goes as before and we get that $k^2 = 0$ for the ground state. The L_m and G_r (for $r \neq 0$) constraints are again trivially satisfied. The G_0 constraint gives us the following;

$$G_0|\mathbf{s};k\rangle_R u_{\mathbf{s}} = \alpha^\mu_0 \psi_{\mu 0}|\mathbf{s};k\rangle_R u_{\mathbf{s}} = \sqrt{\alpha'}|\mathbf{s}';k\rangle_R k^\mu \Gamma_{\mu\mathbf{s}'\mathbf{s}} u_{\mathbf{s}} = 0 \Rightarrow k.\Gamma_{\mathbf{s}'\mathbf{s}} u_{\mathbf{s}} = 0$$

which is the massless Dirac equation. Now, since the state is massless, we can go to the frame where $k_0 = k_1$. In that frame, the Dirac equation becomes;

$$(k_0\Gamma^0 + k_1\Gamma^1)u = k_0(\Gamma^0 - \Gamma^0\Gamma^0\Gamma^1)u = -k_0\Gamma^0(\Gamma^0\Gamma^1 - 1)u = -k_0\Gamma^0(2S_0 - 1)u = -2k_0\Gamma^0\left(S_0 - \frac{1}{2}\right)u = 0$$

and thus, only the states with $s_0 = 1/2$ survive. Now, using (21.1.40) with $k = 4$ and $l = 0$, we get the following decomposition for $SO(9,1) \to SO(1,1) \times SO(8)$;

$$\mathbf{16} = (\mathbf{1},\mathbf{8}) \oplus (\mathbf{1}',\mathbf{8}') = \left(\frac{1}{2},\mathbf{8}\right) \oplus \left(-\frac{1}{2},\mathbf{8}'\right)$$

$$\mathbf{16'} = (\mathbf{1},\mathbf{8}') \oplus (\mathbf{1}',\mathbf{8}) = \left(\frac{1}{2},\mathbf{8}'\right) \oplus \left(-\frac{1}{2},\mathbf{8}\right)$$

So the massless Dirac equation allows a vacuum with $SO(8)$ representation of $\mathbf{8}$ with chirality +1 (which is the same as $e^{\pi i F}$ on the vacuum) and a vacuum with $SO(8)$ representation of $\mathbf{8}$ with $e^{\pi i F} = -1$. We will call these sectors R+ and R−. The open string sectors are then as follows;

Sector	$e^{\pi i F}$	m^2	$SO(8)$
NS+	1	0	$\mathbf{8}_v$
NS−	−1	$-1/2\alpha'$	$\mathbf{1}$
R+	1	0	$\mathbf{8}$
R−	−1	0	$\mathbf{8}'$

For the closed string vacuum, we need to have the level matching condition (i.e. the mass shell condition on the left and right movers must give the same mass). It is written as follows (**derive by rescaling the momentum**);

$$\frac{\alpha'}{4}m^2 = N - \nu = \tilde{N} - \tilde{\nu} \tag{10.5.6}$$

Since the mass of NS− vacuum is different from all the other sectors, it can't be coupled with any other sector. So, the only closed string sector that contains NS− is (NS−, NS−). We can make the following table to help us work out the possible closed string sectors;

Sector	N	ν	$N - \nu$	$\alpha' m^2/4$
NS+	1/2	1/2	0	0
NS−	0	1/2	−1/2	−1/2
R+	0	0	0	0
R−	0	0	0	0

So it is very clear that the allowed sectors are (we exclude sectors that are reflections of other sectors - i.e. that can be obtained by flipping left and right sectors -);

$$(\text{NS}-,\text{NS}-), (\text{NS}+,\text{NS}+), (\text{NS}+,\text{R}+), (\text{NS}+,\text{R}-), (\text{R}+,\text{R}+), (\text{R}+,\text{R}-), (\text{R}-,\text{R}-)$$

Among these sectors, (NS−, NS−) has the tachyon. Before writing down the spectrum of the other sectors, we need to work out the following decompositions. $\mathbf{8}_v \times \mathbf{8}_v$ is a tensor representation but can be broken

sector	$So(8)$ rep	decomposition	dimensions
(NS−, NS−)	$\mathbf{1}$	-	-
(NS+, NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	$[0] + (2) + [2]$	**1+28+35**
(NS+, R+)	$\mathbf{8}_v \times \mathbf{8}$	-	**8'+56**
(NS+, R−)	$\mathbf{8}_v \times \mathbf{8}'$	-	**8+56'**
(R+, R+)	$\mathbf{8} \times \mathbf{8}$	$[0] + [2] + [4]_+$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_+$
(R−, R−)	$\mathbf{8}' \times \mathbf{8}'$	$[0] + [2] + [4]_-$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_-$
(R+, R−)	$\mathbf{8} \times \mathbf{8}'$	$[1] + [3]$	**8+56**

Table 1: Spectrum for closed string sectors

down to a traceless symmetric tensor with $8(8+1)/2 - 1 = 35$ components, an anti-symmetric tensor with $8(8-1)/2 = 28$ components and a pure trace with one component (just like the closed bosonic string massless state). So, we have;

$$\mathbf{8}_v \times \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35} \tag{10.5.7}$$

From (21.1.37), we get the following three decompositions for $k = 3$,

$$\mathbf{8} \times \mathbf{8} = [0] + [2] + [4]_+ = \mathbf{1} + \mathbf{28} + \mathbf{35}_+$$

$$\mathbf{8}' \times \mathbf{8}' = [0] + [2] + [4]_- = \mathbf{1} + \mathbf{28} + \mathbf{35}_-$$

$$\mathbf{8} \times \mathbf{8}' = [1] + [3] = \mathbf{8} + \mathbf{56} \tag{10.5.8}$$

where we used the fact that the number of independent components in an antisymmetric tensor of rank r in d dimensions is $\binom{d}{r}$. We chose $d = 8$. Lastly, we talk about $\mathbf{8}_v \times \mathbf{8}$. We can take a general state in this representation and label it as $|i;\mathbf{s}\rangle$. We can now make eight different linear combinations, labeled by $\mathbf{s}'$ as follows;

$$\Gamma^i_{\mathbf{s}'\mathbf{s}}|i;\mathbf{s}\rangle$$

The chirality of this combination is different from $|i;\mathbf{s}\rangle$ which can be shown as follows;

$$\Gamma|i;\mathbf{s}\rangle = \alpha|i;\mathbf{s}\rangle \Rightarrow \Gamma(\Gamma^i_{\mathbf{s}'\mathbf{s}}|i;\mathbf{s}\rangle) = -\Gamma^i_{\mathbf{s}'\mathbf{s}}\Gamma|i;\mathbf{s}\rangle = -\alpha\Gamma^i_{\mathbf{s}'\mathbf{s}}|i;\mathbf{s}\rangle$$

Under the action of $SO(8)$, this combination maps to a similar combination (**derive this**). Therefore, there is an $\mathbf{8}'$ representation on there. The other 56 states make a single representation (**derive this**). So, we have the following;

$$\mathbf{8}_v \times \mathbf{8} = \mathbf{8}' + \mathbf{56}$$

$$\mathbf{8}_v \times \mathbf{8}' = \mathbf{8} + \mathbf{56}' \tag{10.5.9}$$

Now, the closed string sectors are as follows; **(Write BRST quantization)**

10.6 Superstring theories in ten dimensions

Instead of naming the theories with ν and F, we adopt the more convenient numbers (α, F) where $\alpha = 1 - 2\nu$. This gets rid of the fractions in ν and makes α defined up to mod 2 only. In the open string case, we have the following correspondences;

(α, F)	ν	sector
$(0, 1)$	1/2	NS+
$(0, -1)$	1/2	NS−
$(0, 1)$	0	R+
$(0, -1)$	0	R−

For closed string, we label the sectors by four numbers $(\alpha, F, \bar{\alpha}, \bar{F})$. The correspondences are tabulated as follows;

$(\alpha, F, \tilde{\alpha}, \tilde{F})$	sector
$(0,0,0,0)$	(NS+, NS+)
$(0,0,0,1)$	-
$(0,0,1,0)$	(NS+, R+)
$(0,0,1,1)$	(NS+, R−)
$(0,1,0,0)$	-
$(0,1,0,1)$	(NS+, NS−)
$(0,1,1,0)$	-
$(0,1,1,1)$	-
$(1,0,0,0)$	(R+, NS+)
$(1,0,0,1)$	-
$(1,0,1,0)$	(R+, R+)
$(1,0,1,1)$	(R+, R−)
$(1,1,0,0)$	(R−, NS+)
$(1,1,0,1)$	-
$(1,1,1,0)$	(R−, R+)
$(1,1,1,1)$	(R−, R−)

Some of the possibilities aren't connected to an allowed sector because they involve coupling NS− to another sector. Some sectors are reflections of other sectors. So, we still have only seven possible sectors. Now, we impose conditions to make consistent theories.

1- We want operators to be mutually local (i.e. they don't pick up phase when one operator circles around another). All of the operators can't be mutually local because the vertex operator of Ramond vaccum has branch cuts. Consider two operators in $(\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1)$ $(\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2)$ theories. Then the phase acquired by this circling is **(derive this)**;

$$\exp\left[\pi i(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1)\right] \tag{10.6.1}$$

Note that since $\alpha = 0$ for NS sector, if all the α's vanish, then the phase is unity and it makes sense because there are no branch cuts in the NS sector. So, the first consistency condition is;

$$F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_2\tilde{\alpha}_1 \in 2\mathbb{Z} \tag{10.6.2}$$

2- We recall that in the plane, Ramond operators have half-integer modes and NS operators have integer modes. So, for the operator products in the plane, we have the following;

$$\mathrm{R} \times \mathrm{R} = \mathrm{NS} \Rightarrow 1 \times 1 = 0 \; (= 1 + 1 \bmod 2)$$

$$\mathrm{R} \times \mathrm{NS} = \mathrm{R} \Rightarrow 1 \times 0 = 1 \; (= 1 + 0 \bmod 2)$$

$$\mathrm{NS} \times \mathrm{NS} = \mathrm{NS} \Rightarrow 0 \times 0 = 0 \; (= 0 + 0 \bmod 2)$$

where we wrote the same thing in terms of corresponding α values. So, we see that in the operator products, α's just add up (up to mod 2). It is easier to see that the fermion number also adds up and thus, F is also conserved mod 2. Now, we can state our second condition. We want the OPEs to close and thus, if $(\alpha_1, F_1, \tilde{\alpha}_1, \tilde{F}_1)$ and $(\alpha_2, F_2, \tilde{\alpha}_2, \tilde{F}_2)$, then we should also have the following sector;

$$(\alpha_1 + \alpha_2, F_1 + F_2, \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{F}_1 + \tilde{F}_2) \mod 2$$

3- When we study modular invariance (which we will study later), we get these modular connections between NS and R sectors (e.g. partition function connections). This tells us that we should have at least one left and right moving Ramond sector (a theory with NS sectors alone won't be modular invariant).

Now, if we assume that we have at least one NS−R sector, it has to be $(\mathrm{R+}, \mathrm{NS+}) = (1,0,0,0)$ or $(\mathrm{R-}, \mathrm{NS+}) = (1,1,0,0)$ sector or both. The phase in (10.6.1) for these two sectors is as follows;

$$\exp\left[\pi i((0)(1) - (1)(1) - (0)(0) + (0)(0))\right] = -1$$

So, these two sectors can't co-exist in a theory. We should also have NS – R or R – R sectors (or both sectors) according to the third requirement. Since we already have R – NS sector by assumption, we see that if we have an R – R sector, then the operator product goes as follows;

$$\mathrm{R - NS \times R - R = NS - R}$$

So, we will have an NS sector in any case (i.e. whether or not we have an R sector). So, we have the following possibilities;

$$\mathrm{IIB}:(\mathrm{R}+,\mathrm{NS}+)\ \text{with}\ (\mathrm{NS}+,\mathrm{R}+) \Rightarrow (1,0,0,0)\ \text{with}\ (0,0,1,0)$$
$$\mathrm{IIA}:(\mathrm{R}+,\mathrm{NS}+)\ \text{with}\ (\mathrm{NS}+,\mathrm{R}-) \Rightarrow (1,0,0,0)\ \text{with}\ (0,0,1,1)$$
$$\mathrm{IIA}':(\mathrm{R}-,\mathrm{NS}+)\ \text{with}\ (\mathrm{NS}+,\mathrm{R}+) \Rightarrow (1,1,0,0)\ \text{with}\ (0,0,1,0)$$
$$\mathrm{IIB}':(\mathrm{R}-,\mathrm{NS}+)\ \text{with}\ (\mathrm{NS}+,\mathrm{R}-) \Rightarrow (1,1,0,0)\ \text{with}\ (0,0,1,1)$$

where we have put labels on these possibilities. Let's take IIB possibility first. It has $(1,0,0,0)$ and $(0,0,1,0)$ possibilities. Condition 2 requires us to have the following sectors as well.

$$(1,0,0,0)+(1,0,0,0)=(0,0,0,0)\ (\mathrm{mod}\ 2)=(\mathrm{NS}+,\mathrm{NS}+)$$

$$(1,0,0,0)+(0,0,1,0)=(1,0,1,0)\ (\mathrm{mod}\ 2)=(\mathrm{R}+,\mathrm{R}+)$$

So, the full type IIB possibility is as follows;

$$\mathrm{IIB}:\ (\mathrm{R}+,\mathrm{NS}+)\ ,\ (\mathrm{NS}+,\mathrm{R}+)\ ,\ (\mathrm{NS}+,\mathrm{NS}+)\ ,\ (\mathrm{R}+,\mathrm{R}+) \tag{10.6.3}$$

Let's take the possibility IIA now. We already have $(1,0,0,0)$ and $(0,0,1,1)$ there. Closure of OPEs requires us to have the following sectors;

$$(1,0,0,0)+(1,0,0,0)=(0,0,0,0)\ (\mathrm{mod}\ 2)=(\mathrm{NS}+,\mathrm{NS}+)$$

$$(1,0,0,0)+(0,0,1,1)=(1,0,1,1)\ (\mathrm{mod}\ 2)=(\mathrm{R}+,\mathrm{R}-)$$

All the other OPEs close after adding these two sectors (**(check mutual localities)**). So, the full IIA possibility is as follows;

$$\mathrm{IIA}:\ (\mathrm{R}+,\mathrm{NS}+)\ ,\ (\mathrm{NS}+,\mathrm{R}-)\ ,\ (\mathrm{NS}+,\mathrm{NS}+)\ ,\ (\mathrm{R}+,\mathrm{R}-) \tag{10.6.4}$$

Let's take the possibility IIA' now. We already have $(1,1,0,0)$ and $(0,0,1,0)$ there. Closure of OPEs requires us to have the following sectors;

$$(1,1,0,0)+(1,1,0,0)=(0,0,0,0)\ (\mathrm{mod}\ 2)=(\mathrm{NS}+,\mathrm{NS}+)$$

$$(1,1,0,0)+(0,0,1,1)=(1,1,1,0)\ (\mathrm{mod}\ 2)=(\mathrm{R}-,\mathrm{R}+)$$

All the other OPEs close after adding these two sectors. So, the full IIA possibility is as follows;

$$\mathrm{IIA}':\ (\mathrm{R}-,\mathrm{NS}+)\ ,\ (\mathrm{NS}+,\mathrm{R}+)\ ,\ (\mathrm{NS}+,\mathrm{NS}+)\ ,\ (\mathrm{R}-,\mathrm{R}+) \tag{10.6.5}$$

Finally, we consider the possibility IIB' now. We already have $(1,1,0,0)$ and $(0,0,1,1)$ there. Closure of OPEs requires us to have the following sectors;

$$(1,1,0,0)+(1,1,0,0)=(0,0,0,0)\ (\mathrm{mod}\ 2)=(\mathrm{NS}+,\mathrm{NS}+)$$

$$(1,1,0,0)+(0,0,1,1)=(1,1,1,1)\ (\mathrm{mod}\ 2)=(\mathrm{R}-,\mathrm{R}-)$$

All the other OPEs close after adding these two sectors. So, the full IIA possibility is as follows;

$$\mathrm{IIA}:\ (\mathrm{R}-,\mathrm{NS}+)\ ,\ (\mathrm{NS}+,\mathrm{R}-)\ ,\ (\mathrm{NS}+,\mathrm{NS}+)\ ,\ (\mathrm{R}-,\mathrm{R}-) \tag{10.6.6}$$

If we do a spacetime reflection on an axis e.g. $X^2 \to -X^2$, $\psi^2 \to -\psi^2$ $\tilde{\psi}^2 \to -\tilde{\psi}^2$. It flips the sign of S_1 (see (10.2.14)) and thus, it changes the value of F by 1 (see (10.2.15)). Thus, $e^{\pi i F}$ changes the sign as well. **(Why is this argument true for Ramond only?)**. So, IIA and IIA' (as well as IIB and IIB') are related by a spacetime reflection. **(Check that action and constraints are unchanged in this reflection)**.

Now, if we assume that we have no R−NS sector, condition 3 requires us to have (R+, R+) = $(1,0,1,0)$,(R+, R−) = $(1,0,1,1)$, (R−, R+) = $(1,1,1,0)$ or (R−, R−) = $(1,1,1,1)$ sector. Condition 2 requires us to have (NS+, NS+) sector as well as shown below.

$$(1,0,1,0)+(1,0,1,0)=(0,0,0,0)=(\text{NS+},\text{NS+})$$

$$(1,0,1,1)+(1,0,1,1)=(0,0,0,0)=(\text{NS+},\text{NS+})$$

$$(1,1,1,0)+(1,1,1,0)=(0,0,0,0)=(\text{NS+},\text{NS+})$$

$$(1,1,1,1)+(1,1,1,1)=(0,0,0,0)=(\text{NS+},\text{NS+})$$

Since all α's and F's vanish for (NS+, NS+), the mutual locality is trivially satisfied but these possibilities are not modular invariant **(give reasons for it, probably partition function)**. The modular invariance is restored if we include (NS−, NS−) sector. Then, the two possibilities are as follows;

$$0\text{A}: \ (\text{NS+},\text{NS+}),(\text{R+},\text{R}-),(\text{R}-,\text{R+}),(\text{NS}-,\text{NS}-) \tag{10.6.7}$$

$$0\text{B}: \ (\text{NS+},\text{NS+}),(\text{R+},\text{R+}),(\text{R}-,\text{R}-),(\text{NS}-,\text{NS}-) \tag{10.6.8}$$

These theories obviously have tachyons but they are modular invariant. For IIB theory, all chiralities are +1 and thus, we have:

$$\text{IIB}: \ e^{\pi i F}=e^{\pi i\tilde{F}}=1 \tag{10.6.9}$$

For IIA theory, the left-handed chiralities are positive but the right-handed chiralities are +1 for NS sector (for which $\tilde{\alpha}=0$) and −1 for R sector (for which $\tilde{\alpha}=1$). So, the right-handed chiralities are given by $(-1)^{\tilde{\alpha}}$. Thus, have the following;

$$\text{IIA}: \ e^{\pi i F}=1, \ e^{\pi i\tilde{F}}=(-1)^{\tilde{\alpha}} \tag{10.6.10}$$

These projections are known as GSO projections. The 0B theory has same chirality for left and right movers and it has the same kind of periodicity for left and right movers. So, we get;

$$0\text{B}: \ e^{\pi i F}=e^{\pi i\tilde{F}}, \ \ \alpha=\tilde{\alpha} \tag{10.6.11}$$

This projection is called the diagonal GSO projection. Usingm table 1, we can write down the spectrum of IIA and IIB theories.

$$\text{IIA}: \ [0]+[1]+[2]+[3]+(2)+\mathbf{8}'+\mathbf{56}+\mathbf{8}+\mathbf{56}' \tag{10.6.12}$$

$$\text{IIB}: \ [0]^2+[2]^2+(2)+[4]_++\mathbf{8}'^2+\mathbf{56}^2 \tag{10.6.13}$$

(Talk about gravitinos, oriented strings, open strings and other topics) The closed unoriented type I has the spectrum;

$$\text{IA (closed)}: \quad [0]+(2)+[2]+\mathbf{8}+\mathbf{56} \tag{10.6.14}$$

where [2] comes from (R, R) spectrum and not from (NS+, NS+) spectrum. The open string can couple with unoriented closed string only (because it has one gravitino only **include the reason why this matters**) and thus, the spectrum is;

$$\text{IA (closed, open)}: \quad [0]+(2)+[2]+\mathbf{8}+\mathbf{56}+\mathbf{8}+\mathbf{56} \tag{10.6.15}$$

Include chirality reason why the other open string can't appear. **(oriented strings, open strings and other topics)**

10.7 Modular invariance

We can calculate the torus amplitude for superstrings. We have two species i.e. bosons and fermions and thus, there will be a relative negative sign for the spacetime fermions (**find justification as the negative sign was coming in energy density**). So, we put $d=10$ in (7.3.1) and use the following facts;

$$q\bar{q}=e^{-4\pi\tau_2}\Rightarrow e^{-\pi\tau_2\alpha' k^2}=(q\bar{q})^{\alpha' k^2/4}$$

Moreover,

$$m_i^2=\frac{4}{\alpha'}(h_i-1)=\frac{4}{\alpha'}(\bar{h}_i-1)$$

So, using these two facts, (7.3.1) becomes (for $d = 10$);

$$Z_{T^2} = V_{10} \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \int \frac{d^{10}k}{(2\pi)^{10}} \sum_{i \in \mathcal{H}_\perp} (-1)^{\mathbf{F}_i} q^{\alpha'(k^2+m_i^2)/4} \bar{q}^{\alpha'(k^2+m_i^2)/4}$$

We now have to sum over all possible states. The states are labelled by occupancy numbers of the bosonic modes and fermionic modes. So, we have to add over all possible modes. We also have to do the k integration. We saw in chapter 7 that the contribution from each X is as follows;

$$Z_X(\tau) = (4\pi^2\alpha'\tau_2)^{-1/2}|\eta(\tau)|^2$$

So, the total bosonic contribution will be Z_X^8 (because there are eight transverse directions). However, the k^0 and k^1 directions are still undone, and doing them will give an additional factor of $i(4\pi^2\alpha'\tau_2)^{-1}$ (i comes because we need to Wick rotate $k^0 \to ik^0$).
For the fermionic sector, we need to consider general periodicities. We can do the calculation for periodicities much more general than $\nu = 0$ (R sector) or $\nu = 1/2$ (NS sector). Such periodicities were considered in subsection 10.3 when deriving the vertex operators for R vacuum. Recall that $\alpha = 1 - 2\nu$ and then, the following facts are relevant.
The ground state is;

$$|0\rangle_\nu,\ 0 \le \nu \le 1 \Rightarrow 0 \le \alpha \le 1$$

and the creation operators are as follows;

$$\psi_{n+\nu},\ \tilde{\psi}_{n+1-\nu},\ n = -1, -2, ... \text{ or } \psi_{-n+(1-\alpha)/2}, \tilde{\psi}_{-n+1-(1-\alpha)/2} = \tilde{\psi}_{-n+(1+\alpha)/2},\ n = 1, 2, ...$$

The weight of the ground state is as follows;

$$h_\nu = \frac{1}{2}\left(\nu - \frac{1}{2}\right)^2 = \frac{\alpha^2}{8}$$

The contribution of these two fermions to the conformal weight (or the hamiltonian as hamiltonian is $L_0 + \bar{L}_0$) is as follows;

$$\frac{\alpha^2}{8} + \sum_{m=1}^{\infty}\left(m - \frac{1-\alpha}{2}\right)N_m + \sum_{m=1}^{\infty}\left(m - \frac{1+\alpha}{2}\right)\tilde{N}_m + \text{zero point}$$

where N_m is the occupancy number of $\psi_{-m+(1-\alpha)/2}$ and $\tilde{N}_m$ is the occupancy number of $\tilde{\psi}_{-m+(1+\alpha)/2}$. Now, we have to determine the zero-point energy. The combination;

$$\frac{\alpha^2}{8} + \text{ zero point}$$

should be $2 \times (-1/48) = -1/24$ when $\alpha = 0$ because in this case, the fermions will be in the NS sector (i.e. they are anti-periodic fermions) and since they are two in number, we have a factor of 2. So, zero point should be $-1/24$. So, the total contribution to the hamiltonian by these two fermions is;

$$\frac{\alpha^2}{8} + \sum_{m=1}^{\infty}\left(m - \frac{1-\alpha}{2}\right)N_m + \sum_{m=1}^{\infty}\left(m - \frac{1+\alpha}{2}\right)\tilde{N}_m - \frac{1}{24}$$

$$= \sum_{m=1}^{\infty}\left(m - \frac{1-\alpha}{2}\right)N_m + \sum_{m=1}^{\infty}\left(m - \frac{1+\alpha}{2}\right)\tilde{N}_m + \frac{3\alpha^2 - 1}{24}$$

To take the trace, we proceed as follows;

$$\mathrm{Tr}_\alpha(q^H) = q^{(3\alpha^2-1)/24} \prod_{m=1}^{\infty} \sum_{N_m, \tilde{N}_m = 0}^{1} q^{\left(m - \frac{1-\alpha}{2}\right)\tilde{N}_m} q^{\left(m - \frac{1+\alpha}{2}\right)\tilde{N}_m} = q^{(3\alpha^2-1)/24} \prod_{m=1}^{\infty}\left(1 + q^{m-(1-\alpha)/2}\right)\left(1 + q^{m-(1+\alpha)/2}\right)$$

The occupation numbers are only 0 and 1 because these are fermions. Now we want to define the fermion number for ψ and $\tilde{\psi}$. Since ψ's and $\tilde{\psi}$'s are of the following form in the bosonization;

$$\psi^a = \frac{1}{\sqrt{2}}\left(\psi^{2a} + i\psi^{2a+1}\right),\ \tilde{\psi}^a = \frac{1}{\sqrt{2}}\left(\psi^{2a} - i\psi^{2a+1}\right),\ \ a \in \{1, 2, 3, 4\}$$

we see using (10.2.16), we see that F gives a fermion number +1 for ψ and -1 for $\tilde{\psi}$. In the bosonization picture, it is the momentum for H and the ground state charge is $\alpha/2$ (**derive this**). So, Q mod 2 should be F. One derivation might be as follows. Define $Q = -i(\partial H(z))_{-1}$ and then, we have;

$$-i\partial H(z)|0\rangle_\nu = -i\partial H(z)\exp\left(i\left(-\nu+\frac{1}{2}\right)\mathcal{H}(0)\right)|1\rangle = -i\partial H(z)\exp\left(-i\frac{\alpha}{2}\mathcal{H}(0)\right)|1\rangle$$

$$\sim \frac{\alpha}{2z}\exp\left(-i\frac{\alpha}{2}\mathcal{H}(0)\right)|1\rangle + ... = \frac{\alpha}{2z}|0\rangle_\nu + ...$$

where we used the OPE between $-i\partial H$ and the H exponential. This OPE can be converted to the following commutation relation;

$$\left[(\partial H)_m, \left[\exp\left(-i\frac{\alpha}{2}\mathcal{H}\right)\right]_n\right] = \frac{\alpha}{2}\left[\exp\left(-i\frac{\alpha}{2}\mathcal{H}\right)\right]_{m+n}$$

(**There is a problem with the subscripts when calculating the charge. Solve this**). Using Q, we can define a general partition function as follows;

$$Z^\alpha_\beta(\tau) = \mathrm{Tr}_\alpha\left(q^H e^{\pi i\beta Q}\right)$$

This partition function will have two differences. Firstly, we will have an additional factor because of the vacuum charge under Q. Secondly, it will have an additional factor in the terms that have a fermion occupancy number equal to 1. So, we will have the following;

$$Z^\alpha_\beta(\tau) = q^{(3\alpha^2-1)/24}e^{\pi i\alpha\beta/2}\prod_{m=1}^{\infty}\left[1+e^{\pi i\beta}q^{m-(1-\alpha)/2}\right]\left[1+e^{-\pi i\beta}q^{m-(1+\alpha)/2}\right] \tag{10.7.1}$$

We can write this in a compact form. Using the generalized theta functions, we have;

$$\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(0,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{\alpha}{2}\right)^2\tau + \pi i\beta\left(n+\frac{\alpha}{2}\right)} = e^{\pi i\alpha^2\tau/4}e^{\pi i\alpha\beta/2}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}e^{\pi i n(\alpha\tau+\beta)}$$

$$= q^{\alpha^2/8}e^{\pi i\alpha\beta/2}\sum_{n\in\mathbb{Z}} q^{n^2/2}e^{\pi i n(\alpha\tau+\beta)} = q^{\alpha^2/8}e^{\pi i\alpha\beta/2}\prod_{m=1}^{\infty}(1-q^m)(1+q^{m-1/2}e^{\pi i(\alpha\tau+\beta)})(1+q^{m-1/2}e^{-\pi i(\alpha\tau+\beta)})$$

$$= q^{\alpha^2/8-1/24}e^{\pi i\alpha\beta/2}\eta(\tau)\prod_{m=1}^{\infty}(1+q^{m-1/2}e^{\pi i(\alpha\tau+\beta)})(1+q^{m-1/2}e^{-\pi i(\alpha\tau+\beta)})$$

$$= q^{(3\alpha^2-1)/24}e^{\pi i\alpha\beta/2}\eta(\tau)\prod_{m=1}^{\infty}(1+q^{m-(1-\alpha)/2}e^{\pi i\beta})(1+q^{m-(1+\alpha)/2}e^{-\pi i\beta}) = \eta(\tau)Z^\alpha_\beta(\tau)$$

$$\Rightarrow Z^\alpha_\beta(\tau) = \frac{1}{\eta(\tau)}\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(0,\tau)$$

Now, the four types of partition functions relevant to superstrings are as follows;

$$Z^0_0(\tau) = \mathrm{Tr}_{\mathrm{NS}}(q^H),\ Z^0_1(\tau) = \mathrm{Tr}_{\mathrm{NS}}\left(e^{\pi iF}q^H\right),\ Z^1_0(\tau) = \mathrm{Tr}_{\mathrm{R}}(q^H),\ Z^1_1(\tau) = \mathrm{Tr}_{\mathrm{R}}\left(e^{\pi iF}q^H\right)$$

The total partition function for all fermions is thus given as follows;

$$Z^\pm_\psi(\tau) = \frac{1}{2}\left[Z^0_0(\tau)^4 - Z^0_1(\tau)^4 - Z^1_0(\tau)^4 \pm Z^1_1(\tau)^4\right]$$

(**These minus signs look dubious. They should have appeared inside the partition functions and the fourth power should eliminate them. The projection thing also isn't clear**). The ± refers to $e^{\pi iF} = \pm1$ projection. In type IIA theory, $e^{\pi iF}$ in the right handed sector is -1 and thus, we will have Z^{-*}_ψ and in type IIB theory, $e^{\pi iF}$ in the right handed sector is +1 and thus, we will have Z^{+*}_ψ. So, the total partition function becomes;

$$Z_{T^2} = V_{10}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}\frac{i}{4\pi i\alpha'\tau_2}Z^8_X Z^+_\psi Z^{\pm*}_\psi$$

The + sign is for IIB theory and the − sign is for IIA theory. We now check the modular invariance of this partition function. We see that under T transformation;

$$d^2\tau \to d^2\tau,\ \tau_2 \to \tau_2 \Rightarrow \frac{d^2\tau}{\tau_2^2} \to \frac{d^2\tau}{\tau_2^2}$$

Under S transformation, we have;

$$d\left(-\frac{1}{\tau}\right) = \frac{d\tau}{\tau^2},\ \ d\left(-\frac{1}{\bar{\tau}}\right) = \frac{d\bar{\tau}}{\bar{\tau}^2},\ \ \tau_2 \to \frac{\tau_2}{|\tau|^2},\ \ \bar{\tau}_2 \to \frac{\bar{\tau}_2}{|\tau|^2} \Rightarrow \frac{d^2\tau}{\tau_2^2} \to \frac{d^2\tau}{\tau_2^2}$$

Thus, the measure is modular invariant. Moreover, Z_X is modular invariant as we already know. For Z_ψ, note that Z^α_β is the partition function that arises in the following boundary conditions along the two directions of the torus (**motivate that this is the correct path integral**);

$$\psi(w + 2\pi) = -e^{-\pi i\alpha}\psi(w),\ \psi(w + 2\pi\tau) = -e^{-\pi i\beta}\psi(w) \tag{10.7.2}$$

This implies the following;

$$\psi\left(w + 2\pi(\tau+1)\right) = \psi\left(w + 2\pi\tau + 2\pi\right) = -e^{-\pi i\alpha}\psi\left(w + 2\pi\tau\right) = e^{-\pi i(\alpha+\beta)}\psi\left(w\right) = -e^{-\pi i(\alpha+\beta-1)}\psi\left(w\right)$$

This is just the second boundary condition in (10.7.2) with the following changes;

$$\tau \to \tau + 1,\ \ \beta \to \alpha + \beta - 1$$

So, it seems that the following is true;

$$Z^\alpha_\beta(\tau) = \xi Z^\alpha_{\alpha+\beta-1}(\tau + 1)$$

where ξ can be a proportionality constant. We will see that this is a phase and it is not zero (Polchinski says that naively, $\xi = 1$ but then he proves that $\xi \neq 1$. So, I am inserting ξ from the start). Therefore, we see that sending $\tau \to \tau + 1$ gives us another partition function. We can write the above equation as follows;

$$Z^\alpha_\beta(\tau + 1) = \xi^{-1} Z^\alpha_{\beta-\alpha+1}(\tau)$$

which explicitly gives the T transformation of Z^α_β. Now, we turn towards S transformation. Define $w' = w/\tau$ and $\psi'(w') = \psi(w)$. Then in terms of ψ' the boundary conditions become;

$$\psi'(w' + 2\pi) = \psi'\left(\frac{w}{\tau} + 2\pi\right) = \psi'\left(\frac{w + 2\pi\tau}{\tau}\right) = \psi\left(w + 2\pi\tau\right) = -e^{-\pi i\beta}\psi\left(w\right) = -e^{-\pi i\beta}\psi'\left(w'\right)$$

$$\psi'\left(w' - \frac{2\pi}{\tau}\right) = \psi'\left(\frac{w}{\tau} - \frac{2\pi}{\tau}\right) = \psi'\left(\frac{w - 2\pi}{\tau}\right) = \psi(w - 2\pi) = -e^{i\pi\alpha}\psi(w) = -e^{i\pi\alpha}\psi'(w')$$

These conditions should give us $Z^\alpha_\beta(\tau)$ but with $\beta \to -\alpha, \alpha \to \beta, \tau \to -1/\tau$. So, we see that we should have;

$$Z^\alpha_\beta(\tau) = \zeta Z^\beta_{-\alpha}(-1/\tau)$$

where ζ is again a constant. Since ζ is independent of τ, we can set $\tau = i$ and then, $\tau = -1/\tau$. Thus, we can derive the following;

$$Z^\alpha_\beta(i) = \zeta Z^\beta_{-\alpha}(i) = \zeta^2 Z^{-\alpha}_{-\beta}(i) = \zeta^3 Z^{-\beta}_{\alpha}(i) = \zeta^4 Z^\alpha_\beta(i) \Rightarrow \zeta^4 = 1$$

So, we see that $\zeta^4 = 1$. We will see that $\zeta = 1$. It can be seen from the following calculation;

$$Z^\beta_{-\alpha}(-1/\tau) = \frac{1}{\eta(-1/\tau)}\vartheta\begin{bmatrix}\beta/2\\ -\alpha/2\end{bmatrix} = \frac{1}{\sqrt{-i\tau}\eta(\tau)}\sum_{n\in\mathbb{Z}}\exp\left[-\frac{\pi i}{\tau}\left(n + \frac{\beta}{2}\right)^2 - \pi i\left(n + \frac{\beta}{2}\right)\alpha\right]$$

$$= \frac{1}{\sqrt{-i\tau}\eta(\tau)}\exp\left(-\frac{\pi i}{4\tau}(\beta^2 + 2\alpha\beta\tau)\right)\sum_{n\in\mathbb{Z}}\exp\left[-\frac{\pi i n^2}{\tau} - \pi i\left(\frac{\beta}{\tau} + \alpha\right)n\right]$$

$$= \frac{\sqrt{-i\tau}}{\sqrt{-i\tau}\eta(\tau)} \exp\left(-\frac{\pi i}{4\tau}(\beta^2 + 2\alpha\beta\tau)\right) \sum_{n\in\mathbb{Z}} \exp\left[i\pi\tau\left(n - \frac{1}{2}\left(\frac{\beta}{\tau} + \alpha\right)\right)^2\right]$$

$$= \frac{1}{\eta(\tau)} \exp\left(-\frac{\pi i}{4\tau}(\beta^2 + 2\alpha\beta\tau)\right) \sum_{n\in\mathbb{Z}} \exp\left[i\pi\tau n^2 - i\pi\tau n\left(\frac{\beta}{\tau} + \alpha\right) + \frac{i\pi\tau}{4}\left(\frac{\beta}{\tau} + \alpha\right)^2\right]$$

$$= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \exp\left[i\pi\tau\left(n^2 - n\alpha + \frac{\alpha^2}{4}\right) - 2\pi i n\frac{\beta}{2}\right] = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \exp\left[i\pi\tau\left(n - \frac{\alpha}{2}\right)^2 - 2\pi i n\frac{\beta}{2}\right]$$

$$= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \exp\left[i\pi\tau\left(n + \frac{\alpha}{2}\right)^2 + 2\pi i n\frac{\beta}{2}\right] = Z^{\alpha}_{\beta}(\tau)$$

where in the last step, we changed the summation variable i.e. $n \to -n$. Also note from this calculation that $Z^{-\alpha}_{-\beta}(\tau) = Z^{\alpha}_{\beta}(\tau)$. Also, we have shown that $\zeta = 1$. To find ξ, we need (10.7.1) to calculate $Z^{\alpha}_{\alpha+\beta-1}(\tau+1)$. We establish the following identities first. We will use q_2 for $e^{2\pi i(\tau+1)}$. We start with the following results;

$$e^{\pi i(\beta+\alpha-1)} q_2^{m-(1-\alpha)/2} = e^{\pi i(\beta+\alpha-1)} q^{m-(1-\alpha)/2} e^{-\pi i(1-\alpha)} = e^{\pi i(\beta+2\alpha)} q^{m-(1-\alpha)/2}$$

$$e^{-\pi i(\beta+\alpha-1)} q_2^{m-(1+\alpha)/2} = e^{-\pi i(\beta+\alpha-1)} q^{m-(1+\alpha)/2} e^{-\pi i(1+\alpha)} = e^{-\pi i(\beta+2\alpha)} q^{m-(1+\alpha)/2}$$

$$q_2^{(3\alpha^2-1)/24} = q^{(3\alpha^2-1)/24} e^{\pi i(3\alpha^2-1)/12}$$

$$e^{\pi i\alpha(\beta+\alpha-1)/2} = e^{\pi i\alpha\beta/2} e^{\pi i\alpha(\alpha-1)/2} = e^{\pi i\alpha(\beta+2\alpha)/2} e^{-\pi i\alpha(\alpha+1)/2}$$

But this gives;

$$Z^{\alpha}_{\alpha+\beta-1}(\tau+1) = \exp\left[\pi i\left(\frac{3\alpha^2-1}{12}\right) - \frac{\pi i\alpha(\alpha+1)}{2}\right] Z^{\alpha}_{\beta+2\alpha}(\tau)$$

This isn't the right transformation. We can now show that we can always come to $\alpha = 1$ if we start from $\alpha = 0$ (for $\beta = 0, 1$ which are present in $Z^{\pm}_{\psi}$) and thus, the Ramond sector should be present due to modular invariance. We use the following two transformations of the partition functions;

$$Z^{\alpha}_{\beta} \xrightarrow{T} Z^{\alpha}_{\beta+\alpha-1}, \quad Z^{\alpha}_{\beta} \xrightarrow{S} Z^{\beta}_{-\alpha}$$

Moreover, we also use the previously noted result that $Z^{\alpha}_{\beta}(\tau) = Z^{-\alpha}_{-\beta}(\tau)$. The required transformations are as follows;

$$Z^0_1 \xrightarrow{S} Z^1_0$$

$$Z^0_0 \xrightarrow{T} Z^0_{-1} \xrightarrow{S} Z^{-1}_0 = Z^1_0$$

The second transformation can be done as following as well;

$$Z^0_0 \xrightarrow{T} Z^0_{-1} \xrightarrow{S} Z^{-1}_0 \xrightarrow{S} Z^0_1 \xrightarrow{S} Z^1_0$$

(**What about general beta?**). Observe that Z^0_0 and Z^0_1 are both brought to Z^1_0 and since the modular transformations are invertible, Z^1_0 can be brought to Z^0_1 or Z^0_0. However, none of Z^0_0 or Z^0_1 can be brought to Z^1_1 and thus, we choose at least one Ramond sector to be in the left and right sectors because choosing one NS sector won't be enough. We can also see the following;

$$Z^1_1 \xrightarrow{T} Z^1_1, \quad Z^1_1 \xrightarrow{S} Z^1_{-1} \begin{cases} \xrightarrow{T} Z^1_{-1} \\ \xrightarrow{S} Z^{-1}_{-1} = Z^1_1 \end{cases}$$

So, Z^1_1 transforms into itself under all modular transformations (up to phases). **(What if we make a partition function just out of this?)**. We now see how $Z^{\pm}_{\psi}$ transforms under modular transformations. Let's do the T transformation first;

$$Z^{\pm}_{\psi}(\tau+1) = \frac{1}{2}\left[Z^0_0(\tau+1)^4 - Z^0_1(\tau+1)^4 - Z^1_0(\tau+1)^4 \mp Z^1_1(\tau+1)^4\right]$$

$$= \frac{1}{2}\left[e^{-\pi i/3} Z^0_1(\tau)^4 - e^{-\pi i/3} Z^0_2(\tau)^4 - e^{2\pi i/3} Z^1_0(\tau)^4 \mp e^{2\pi i/3} Z^1_1(\tau)^4\right]$$

$$= \frac{1}{2}\left[-e^{2\pi i/3}Z_1^0(\tau)^4 + e^{2\pi i/3}Z_2^0(\tau)^4 - e^{2\pi i/3}Z_0^1(\tau)^4 \mp e^{2\pi i/3}Z_1^1(\tau)^4\right]$$

$$= \frac{1}{2}e^{2\pi i/3}\left[Z_2^0(\tau)^4 - Z_1^0(\tau)^4 - Z_0^1(\tau)^4 \mp Z_1^1(\tau)^4\right]$$

Now using (10.7.1), we see that;

$$Z_{\beta+2}^\alpha(\tau) = e^{\pi i\alpha}Z_\beta^\alpha(\tau) \Rightarrow Z_{\beta+2}^0(\tau) = Z_\beta^0(\tau) \Rightarrow Z_2^0(\tau) = Z_0^0(\tau)$$

So, we get the following;

$$Z_\psi^\pm(\tau+1) = \frac{1}{2}e^{2\pi i/3}\left[Z_0^0(\tau)^4 - Z_1^0(\tau)^4 - Z_0^1(\tau)^4 \mp Z_1^1(\tau)^4\right] = e^{2\pi i/3}Z_\psi^\pm(\tau)$$

$$\Rightarrow Z_\psi(\tau+1)Z_\psi^*(\bar\tau+1) = Z_\psi(\tau)Z_\psi^*(\bar\tau)$$

for all subscripts of Z_ψ. For S transformations, we proceed as follows;

$$Z_\psi^\pm(-1/\tau) = \frac{1}{2}\left[Z_0^0(-1/\tau)^4 - Z_1^0(-1/\tau)^4 - Z_0^1(-1/\tau)^4 \mp Z_1^1(-1/\tau)^4\right]$$

$$= \frac{1}{2}\left[Z_0^0(\tau)^4 - Z_0^{-1}(\tau)^4 - Z_1^0(\tau)^4 \mp Z_1^{-1}(\tau)^4\right]$$

Now, using the fact that $Z_0^{-1}(\tau) = Z_0^1(\tau)$ and $Z_1^{-1}(\tau) = e^{\pi i}Z_{-1}^{-1}(\tau) = -Z_{-1}^{-1}(\tau) = -Z_1^1(\tau)$, we get;

$$Z_\psi^\pm(-1/\tau) = \frac{1}{2}\left[Z_0^0(\tau)^4 - Z_0^1(\tau)^4 - Z_1^0(\tau)^4 \mp Z_1^1(\tau)^4\right] = Z_\psi^\pm(\tau)$$

So, $Z_\psi^\pm(\tau)$ is invariant under S transformation.
It is much easier to show that the $Z_\psi^\pm$ is modular invariant by showing that it vanishes. Firstly, we see that;

$$Z_0^0(\tau)^4 - Z_0^1(\tau)^4 - Z_1^0(\tau)^4 = \frac{1}{\eta(\tau)^4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau)^4\right)$$

$$= \frac{1}{\eta(\tau)^4}\left(\vartheta_3(0|\tau)^4 - \vartheta_4(0|\tau)^4 - \vartheta_2(0|\tau)^4\right) = 0$$

where we used (7.2.2). Moreover, using (7.2.3), we can deduce that $Z_1^1(\tau) = 0$.

10.7.1 More on $c = 1$ CFT

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10.8 Divergences of type I theory

10.8.1 The cylinder

We need to compute the cylinder, Mobius strip, and Klein bottle partition functions and their massless divergences. For the cylinder partition function, just write the bosonic part, set $d = 10$, and add an additional 1/2 because of the projection operator. Moreover, for the fermionic part, multiply Z_ψ^+. So, we get;

$$Z_{C_2}^{II} = iV_{10}n^2\int_0^\infty \frac{dt}{4t}(8\pi\alpha' t)^{-5}\eta(it)^{-8}.\frac{1}{2}\left[Z_0^0(it)^4 - Z_1^0(it)^4 - Z_0^1(it)^4 - Z_1^1(it)^4\right]$$

$$= iV_{10}n^2\int_0^\infty \frac{dt}{8t}(8\pi\alpha' t)^{-5}\eta(it)^{-8}\left[Z_0^0(it)^4 - Z_0^1(it)^4\right] + iV_{10}n^2\int_0^\infty \frac{dt}{8t}(8\pi\alpha' t)^{-5}\eta(it)^{-8}\left[-Z_1^0(it)^4 - Z_1^1(it)^4\right]$$

$$= Z_{C_2,0} + Z_{C_2,1}$$

where $Z_{C_2,0}$ and $Z_{C_2,1}$ equal to the two terms shown above. Since there is no $e^{\pi iF}$ in the definition of $Z_{C_2,0}$, the fermions appearing in this expression are anti-periodic (**this is related to a related problem in chapter 7. Explain this thing.**). So, $Z_{C_1,0}$ comes from the NS – NS sector. We now focus on $Z_{C_2,0}$

because $Z_{C_2,0} = -Z_{C_2,1}$ due to supersymmetry (**Add more detail if possible**). The modular transformation of $Z_{C_2,0}$ is done as follows;

$$-iV_{10}n^2 \int_0^\infty \frac{d(\pi/s)}{8(\pi/s)} \left(\frac{8\pi^3\alpha'}{s}\right)^{-5} \eta\left(\frac{i\pi}{s}\right)^{-8} \left[Z_0^0\left(\frac{i\pi}{s}\right)^4 - Z_0^1\left(\frac{i\pi}{s}\right)^4\right]$$

$$= -iV_{10}n^2 \int_0^\infty \frac{d(\pi/s)}{8(\pi/s)} \left(\frac{8\pi^3\alpha'}{s}\right)^{-5} \left(\frac{\pi}{s}\right)^4 \eta\left(\frac{is}{\pi}\right)^{-8} \left[Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_{-1}^0\left(\frac{is}{\pi}\right)^4\right]$$

$$= \frac{iV_{10}n^2}{8\pi(8\pi^2\alpha')^5} \int_0^\infty ds\, \eta\left(\frac{is}{\pi}\right)^{-8} \left[Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_1^0\left(\frac{is}{\pi}\right)^4\right]$$

where we used the modular transformations of η and Z_β^α and we also used the fact that $Z_{\beta+2}^0 = Z_\beta^0 \Rightarrow Z_{-1}^0 = Z_1^0$. Now, we extract the massless divergence from this partition function. We use the following fact;

$$\eta(is/\pi)^{-8} = \left(e^{-s/12}\right)^{-8}(1-e^{-2s}+...)^{-8} = e^{2s/3}(1+8e^{-2s}+...) = e^{2s/3}+8e^{-4s/3}+... = e^{2s/3}+\mathcal{O}(e^{-4s/3}) \quad (10.8.1)$$

Moreover, we have;

$$Z_0^0\left(\frac{is}{\pi}\right)^4 - Z_1^0\left(\frac{is}{\pi}\right)^4 = \frac{1}{\eta(is/\pi)^4}\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,is/\pi)^4 - \frac{1}{\eta(is/\pi)^4}\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(0,is/\pi)^4$$

$$= \frac{\vartheta_3(0,is/\pi)^4}{\eta(is/\pi)^4} - \frac{\vartheta_4(0,is/\pi)^4}{\eta(is/\pi)^4} = \frac{\vartheta_2(0,is/\pi)^4}{\eta(is/\pi)^4}$$

where we used (7.2.2). Now, we expand $\vartheta_2(0,\tau)$ as follows;

$$\vartheta_2(0,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} = q^{1/8} + \sum_{n=1}^\infty q^{\frac{1}{2}(n-\frac{1}{2})^2} + \sum_{n=1}^\infty q^{\frac{1}{2}(n+\frac{1}{2})^2} = 2q^{1/8} + \sum_{n=2}^\infty q^{\frac{1}{2}(n-\frac{1}{2})^2} + \sum_{n=1}^\infty q^{\frac{1}{2}(n-\frac{1}{2})^2}$$

$$= 2q^{1/8} + 2\sum_{n=1}^\infty q^{\frac{1}{2}(n+\frac{1}{2})^2}$$

where we merged the two sums by shifting the index $n \to n+1$ in the first sum. Continuing, we get;

$$\vartheta_2(0,\tau) = 2e^{\pi i\tau/4} + 2\sum_{n=1}^\infty e^{\pi i\tau(n+\frac{1}{2})^2} = 2e^{\pi i\tau/4} + 2e^{9\pi i\tau/4} + ...$$

$$\Rightarrow \vartheta_2(0,is/\pi) = 2e^{-s/4} + 2e^{-9s/4} + ... = 2e^{-s/4}\left(1+e^{-4s}+...\right)$$

$$\Rightarrow \vartheta_2(0,is/\pi)^4 = 16e^{-s}\left(1+e^{-4s}+...\right)^4 = 16e^{-s}\left(1+4e^{-4s}+...\right) = 16e^{-s}+64e^{-4s}+...$$

Moreover, we have;

$$\eta(is/\pi)^{-4} = e^{s/3}\left(1-e^{-2s}+...\right)^{-4} = e^{s/3}+4e^{-5s/3}+...$$

Therefore, we get;

$$\frac{\vartheta(0,is/\pi)^4}{\eta(is/\pi)^4} = \left(e^{s/3}+4e^{-5s/3}+...\right)\left(16e^{-s}+64e^{-4s}+...\right) = 16e^{-2s/3}+\mathcal{O}(e^{-8s/3}) \quad (10.8.2)$$

Using (10.8.1) and (10.8.2), we get;

$$\eta(is/\pi)^{-8}\left[Z_0^0(is/\pi)^4 - Z_1^0(is/\pi)^4\right] = \eta(is/\pi)^{-8}\frac{\vartheta(0,is/\pi)^4}{\eta(is/\pi)^4}$$

$$= \left(e^{2s/3}+\mathcal{O}(e^{-4s/3})\right)\left(16e^{-2s/3}+\mathcal{O}(e^{-8s/3})\right) = 16+\mathcal{O}(e^{-2s}) \quad (10.8.3)$$

So, the massless divergence from the cylinder partition function becomes;

$$Z_{C_2} = \frac{16iV_{10}n^2}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds$$

(**Write about the paradox and the Ramond Ramond field resolution**).

10.8.2 The Klein bottle

To do the Klein bottle partition function, we see that we have already the bosonic part. We can convert that to $d = 10$ case and then, we have;

$$Z_{K_2}^{\text{bosonic}} = iV_{10}\int_0^\infty \frac{dt}{4t}(4\pi\alpha' t)^{-5}\eta(2it)^{-8}$$

However, we have to derive the fermionic part. In the torus case, we defined a general partition function Z_β^α which has $e^{i\beta Q} = e^{i\beta F}$ in the trace. The partition function corresponds to the following boundary conditions (recall that we are discussing the torus case tight now);

$$\psi(w+2\pi) = -e^{-\pi i\alpha}\psi(w), \quad \psi(w+2\pi\tau) = -e^{-\pi i\beta}\psi(w)$$

However, in the Klein bottle case, $\psi(w+2\pi it)$ can not be proportional to $\psi(w)$ because in the Klein bottle case, we have $w+2\pi it \sim -\bar{w}$. So, it should be proportional to $\tilde{\psi}(\bar{w})$. Therefore, we will need to insert something in the trace that gives such a boundary condition. Moreover, obviously we need to have Ω and the GSO projection operators in the trace as well. The required object is;

$$R = \Omega\exp\left(\pi i\beta F + \pi i\tilde{\beta}\tilde{F}\right)$$

This gives the following boundary conditions (**derive these**);

$$\psi(w+2\pi it) = -e^{\pi i\beta}\tilde{\psi}(\bar{w}), \;\; \psi(\bar{w}+2\pi it) = -e^{\pi i\tilde{\beta}}\psi(w)$$

This implies;

$$\psi(w+4\pi it) \sim -e^{\pi i\beta}\tilde{\psi}(\bar{w}+4\pi it) = e^{i\pi(\beta+\tilde{\beta})}\psi(w) \tag{10.8.4}$$

This agrees with the fact that for the Klein bottle,

$$w+4\pi it \sim -\bar{w}+2\pi it = -\overline{(w+2\pi it)} = -\overline{(-\bar{w})} = w$$

Since the fundamental region for Klein bottle can be taken to be;

$$0 \le \sigma^1 < \pi, \;\; 0 \le \sigma^2 < 4\pi it$$

The NS – NS sector (**write about why NS-NS or R-R are considered)** corresponds to the case where we have anti-periodic boundary conditions in (10.8.4). This corresponds to $(\beta,\tilde{\beta}) = (0,1)$ or $(1,0)$, which means that $R = \Omega e^{\pi iF}$ or $R = \Omega e^{\pi i\tilde{F}}$. R – R sector corresponds to the case where $(\beta,\tilde{\beta}) = (0,0)$ or $(1,1)$ which gives $R = \Omega$ or $R = \Omega e^{i\pi F + i\pi\tilde{F}}$. The NS – NS and R – R partition functions are then (**derive these**);

$$\text{NS}-\text{NS}: Z_0^0(2it), \;\; \text{R}-\text{R}: -Z_0^1$$

and thus, the full Klein bottle partition function is;

$$Z_{K_2} = iV_{10}\int_0^\infty \frac{dt}{8t}(4\pi^2\alpha' t)^{-5}\eta(2it)^{-8}\left[Z_0^0(2it)^4 - Z_0^1(2it)^4\right]$$

where the minus sign in again due to the spacetime statistics. (**The extra factor of 1/2 is due to GSO projection?**). Now, we do a modular transformation by defining $s = \pi/2t$. We get;

$$\begin{aligned}
Z_{K_2} &= iV_{10}\int_0^\infty \frac{d(\pi/2s)}{8(\pi/2s)}\left(\frac{2\pi^3\alpha'}{s}\right)^{-5}\eta\left(\frac{i\pi}{s}\right)^{-8}\left[Z_0^0(i\pi/s)^4 - Z_0^1(i\pi/s)^4\right] \\
&= \frac{iV_{10}}{8(2\pi^3\alpha')^5}\int_0^\infty \frac{ds}{s}s^5\left(\frac{s}{\pi}\right)^{-4}\eta\left(\frac{is}{\pi}\right)^{-8}\left[Z_0^0(is/\pi) - Z_{-1}^0(is/\pi)\right] \\
&= \frac{i2^{10}V_{10}}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\,\eta\left(\frac{is}{\pi}\right)^{-8}\left[Z_0^0(is/\pi) - Z_1^0(is/\pi)\right]
\end{aligned}$$

where we again used the fact that $Z_{-1}^0 = Z_1^0$. Now, we extract the massless divergence from this partition function. For that, we already have the expansion of the integrand in (10.8.3). So, we have;

$$Z_{K_2} = \frac{i2^{10}V_{10}}{8\pi(8\pi^2\alpha')^5}\int_0^\infty ds\left(16 + \mathcal{O}(e^{-2s})\right)$$

10.8.3 The Mobius strip

Firstly, let's work out the effect of Ω on fermionic fields. We first review the doubling trick for the open string fermions. We have $\psi^\mu(\sigma^1 + i\sigma^2) = \psi^\mu(w)$ and $\tilde{\psi}^\mu(\sigma^1 + i\sigma^2) = \tilde{\psi}^\mu(\bar{w})$. Here, we have $0 \le \sigma^1 \le \pi$. Now, we make an extended fermion that we will call $\Psi^\mu(\sigma^1, \sigma^2)$ (Polchinski calls this ψ^μ as well) which is defined as follows;

$$\Psi^\mu(\sigma^1,\sigma^2) = \Psi^\mu(w) = \begin{cases} \psi^\mu(\sigma^1,\sigma^2) = \psi^\mu(w), & 0 \le \sigma^1 \le \pi \\ \tilde{\psi}^\mu(2\pi - \sigma^1, \sigma^2) = \tilde{\psi}^\mu(2\pi - \bar{w}), & \pi \le \sigma^1 \le 2\pi \end{cases}$$

Now, Ω sends (σ^1, σ^2) to $(\pi - \sigma^1, \sigma^2)$ which sends $w = \sigma^1 + i\sigma^2$ to $\pi - \sigma^1 + i\sigma^2 = \pi - \bar{w}$. It is easily shown that Ψ depends on w only (**include its small proof**). Now, Ω acts as follows;

$$\Omega\Psi^\mu(w)\Omega^{-1} = \Omega\psi^\mu(w)\Omega^{-1} = \tilde{\psi}^\mu(\pi - \bar{w}) = \tilde{\psi}(\pi - \sigma^1, \sigma^2) = \Psi^\mu(\sigma^1 + \pi, \sigma^2) \quad \text{for } 0 \le \sigma^1 \le \pi$$

$$\Omega\Psi^\mu(w)\Omega^{-1} = \Omega\tilde{\psi}^\mu(2\pi - \bar{w})\Omega^{-1} = \tilde{\psi}^\mu(\pi - \bar{w}) = \tilde{\psi}(\pi - \sigma^1, \sigma^2) = \Psi^\mu(\sigma^1 + \pi, \sigma^2) \quad \text{for } \pi \le \sigma^1 \le 2\pi$$

(Complete this. This is so confusing because of multiple domains). In terms of modes, we have (**complete its proof**);

$$\Omega\psi^\mu_r\Omega^{-1} = e^{-\pi i r}\psi^\mu_r$$

In the NS sector, $r \in \mathbb{Z} + 1/2$ and thus, $\Omega^2 = -1$. For the NS vacuum, Ω^2 is the same as $e^{\pi i F}$ i.e. both are -1 (**check that this is true for arbitrary NS states**). Now, the combined Ω and GSO projection operators get the following form in the NS sector;

$$\frac{1+\Omega}{2}\frac{1+e^{\pi i F}}{2} = \frac{1+\Omega}{2}\frac{1+\Omega^2}{2} = \frac{1+\Omega+\Omega^2+\Omega^3}{4}$$

(Complete this calculation)

11 Chapter 11: The Heterotic String

11.1 World sheet supersymmetries

We want to classify all the worldsheet symmetry algebras with the following constraints;

- The spins of the holomorphic current are less than or equal to 2. The higher spin currents (called W algebras) have been used as symmetry currents (**find more about this**) but the constructions (e.g. of the BRST operator) are non-trivial because of the non-linear commutators. Some cases turn out to be special cases of bosonic strings.
- The spin is a multiple of 1/2. If they are not, then the OPE;

$$j(z)j(0) \sim \frac{d_{jj}}{z^{2h}}$$

(where d_{jj} is a constant) will be multi-valued. The string theories that relax this condition are **fractional strings** and it is not clear if these theories exist.

The only spins left then are as follows;

$$h = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

We can make algebras with different number of different types of currents. Before proceeding, we need to calculate the central charge of the ghosts corresponding to a current of spin h (**derive this**). The formula is as follows;

$$c_g = (-1)^{2h+1}(3(2h-1)^2 - 1)$$
$$\Rightarrow c_0 = -2,\ c_{1/2} = -1,\ c_1 = -2,\ c_{3/2} = 11,\ c_2 = -26 \tag{11.1.1}$$

The Jacobi identity allows only the following algebras;

n_0	$n_{1/2}$	n_1	$n_{3/2}$	n_2	c_g
0	0	0	0	1	-26
0	0	0	1	1	$-26+11=-15$
0	0	1	2	1	$-2+2(11)-26=-6$
0	1	3	3	1	$-1+3(-2)+3(11)-26=0$
0	4	7	4	1	$4(-1)+7(-2)+4(11)-26=0$
1	4	6	4	1	$-2+4(-1)+6(-2)+4(11)-26=0$
0	0	3	4	1	$3(-2)+4(11)-26=12$

The first two possibilities are Virasoro and $N=1$ superconformal algebras. The third possibility is called $N=2$ superconformal algebra. After the first three, no algebra has a negative ghost central charge and thus, none of them will have a positive critical dimension. (**Write some details about $N=2$ superconformal algebra**).

11.2 The $SO(32)$ and $E_8 \times E_8$ heterotic strings

The heterotic string appears when we choose the left and right moving algebra to be different. We will study $(N,\tilde{N}) = (0,1)$ heterotic. The other possibilities are $(N,\tilde{N}) = (0,2)$ or $(N,\tilde{N}) = (1,2)$ heterotic (the order is kept ascending because the flipped possibility is realized if we we the flip $z \to \bar{z}$. To have $(N,\tilde{N}) = (0,1)$, we need to have bosonic string on the left moving side and fermionic string on the right moving side. To keep the dimension the same, we need to have the following fields;

$$X_L^\mu(z) \quad (\mu = 0, ..., 9) \quad \text{for left moving side}$$
$$X_R^\mu(\bar{z}),\ \tilde{\psi}^\mu(\bar{z}) \quad (\mu = 0, ..., 9) \quad \text{for right moving side} \tag{11.2.1}$$

However, if we do this, then the central charges for the ghosts are $(c_g, \tilde{c}_g) = (-26, -15)$. The central charge on the right is canceled out by the matter central charge but the central charge on the left will be canceled out only if we put some internal CFT with $(c,\tilde{c}) = (16,0)$ into the game. One way to do this is to introduce 32 fermion fields on the left moving side as follows;

$$X^\mu_L(z) \quad (\mu = 0,...,9),\ \lambda^A(z),\ \ (A = 1,...32) \quad \text{for left moving side}$$

$$X^\mu_R(\bar{z}),\ \tilde{\psi}^\mu(\bar{z}) \quad (\mu = 0,...,9) \quad \text{for right moving side} \tag{11.2.2}$$

This will give us the following action;

$$S = \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu + \lambda^A\bar{\partial}\lambda^A + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right) \tag{11.2.3}$$

Note that there won't be any $T_F(z)$ field now and thus, no fermion constraint on the right moving side. This forbids us to choose timelike directions for any $\lambda^A(z)$. The theory has $SO(1,9)\times SO(32)$ symmetry now where $SO(32)$ is the internal symmetry. The periodicity of $T_B(z)$ (which will contain λ^A's instead of ψ^μ's now) only requires to have the following boundary condition on $\lambda^A(w)$ (notice the w coordinate and hence notice that we are imposing this boundary condition on the cylinder field);

$$\lambda^A(w+2\pi) = \mathcal{O}^{AB}\lambda^B(w),\ \ (\mathcal{O}\in SO(32))$$

The world sheet current for spacetime symmetry is $\mathcal{V}_\mathbf{s}$ (**derive this**). Now, every vertex operator in R sector has $\mathcal{V}_\mathbf{s}$, the OPE of $\mathcal{V}_\mathbf{s}$ with itself should be single-valued. So, we have the following;

$$\mathcal{V}_\mathbf{s}(z)\mathcal{V}_\mathbf{s}(0) \sim e^{-\phi(z)/2}\exp\left[\sum_a s_a H^a(z)\right]e^{-\phi(0)/2}\exp\left[\sum_a s_a H^a(0)\right]$$

$$\sim z^{-1/4}\prod_{j=0}^{4} e^{is_j H^j(z)}e^{is'_j H^j(z)} \sim z^{-\frac{1}{4}}z^{\mathbf{s}.\mathbf{s}'}e^{i\sum_a (s+s')_a H^a(0)}$$

The power of z should be an integer. So, we have the following condition;

$$\mathbf{s}.\mathbf{s}' - \frac{1}{4}\in\mathbb{Z}\ \text{(Ramond sector)}$$

In the NS sector, this condition becomes (**what does it mean to have these spins in NS sector?**);

$$\mathbf{s}.\mathbf{s}' - \frac{1}{2}\in\mathbb{Z}\ \text{(NS sector)}$$

the appearance of $-1/2$ instead of $-1/4$ can be understood because of the appearance of $e^{-\phi(z)}$ in the NS vertex operators instead of $e^{-\phi(z)/2}$. **(Problem with recovering GSO projection from it)**.
(Write about the closure of OPEs and modular invariance).
Now we find the states in the heterotic theory. Let's start with the left-handed sector. This sector has $SO(8)\times SO(32)$ symmetry (SO(8) coming from the transverse X^i's and $SO(32)$ from the λ^A's). The normal ordering constants in the left-handed sector are as follows;

$$\text{NS}: -\frac{8}{24} - \frac{32}{48} = -1,\ \ \text{R}: -\frac{8}{24} + \frac{32}{24} = 1 \tag{11.2.4}$$

Therefore, the R sector has no massless states. So, the transverse Hamiltonian is as follows;

$$H = \frac{\alpha' p^2}{4} + N - 1\ \text{(NS)},\ \ H = \frac{\alpha' p^2}{4} + N + 1\ \text{(R)} \tag{11.2.5}$$

Using the mass shell condition, we see that for $N = 0$, the NS ground state is a tachyon. The first excited states are as follows;

$$\lambda^A_{-1/2}|0\rangle_{\text{NS}}$$

but they are forbidden due to the GSO projection (because F is odd for this state). So, the actual first excited states are as follows;

$$\alpha^i_{-1}|0\rangle_{\text{NS}},\ \ \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{\text{NS}}$$

They satisfy the GSO projection and the mass shell condition implies that these are massless states. The α state lives in $(\mathbf{8}_v, \mathbf{1})$ representation of $SO(8) \times SO(32)$ and λ states lives in $(\mathbf{1}, [2])$ representation of $SO(8) \times SO(32)$. The λ's live in the adjoint representation of $SO(32)$ with the dimension of 496. So, the massless sector lives in $(\mathbf{8}_v, \mathbf{1}) + (\mathbf{1}, \mathbf{496})$.
Now, the right-handed sector (because of the GSO projection), has no tachyon. Moreover, the massless states (like type IIB theory) live in $\mathbf{8}_v + \mathbf{8}$ representation ($\mathbf{8}_v$ comes from the NS sector and $\mathbf{8}$ comes from the R sector). So, the massless states are as follows;

$$[(\mathbf{8}_v, \mathbf{1}) + (\mathbf{1}, \mathbf{496})] \times (\mathbf{8}_v + \mathbf{8}) \tag{11.2.6}$$

These states can be broken down using the results that we derived before. This breakdown can be divided into two parts. They are as follows;

$$(\mathbf{8}_v, \mathbf{1}) \times (\mathbf{8}_v + \mathbf{8}) = (\mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}) \quad \text{(type I supergravity multiplet)} \tag{11.2.7}$$

$$(\mathbf{1}, \mathbf{496}) \times (\mathbf{8}_v + \mathbf{8}) = (\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}, \mathbf{496}) \quad \text{(SO(32) N=1 gauge multiplet)} \tag{11.2.8}$$

(Write about the similarity of this theory and type I theory).
Another heterotic theory is obtained by having the following boundary conditions;

$$\lambda^A(w + 2\pi) = \begin{cases} \eta \lambda^A(w) \ A = 1, ..., 16 \\ \eta' \lambda^A(w) \ A = 17, ..., 32 \end{cases} \tag{11.2.9}$$

where η and η' can be ± 1. So, we have four $\mathrm{NS-NS'}, \mathrm{NS-R'}, \mathrm{R-NS'}$ and $\mathrm{R-R'}$ sectors in the left-moving theory. In the left sector, the GSO projection is applied to both subsectors separately. **(Write about the closure of OPE and modular invariance)**.
For the $\mathrm{NS-NS'}$ sector, the normal ordering constant is still -1 and thus, we still have a tachyon in the left sector and the first allowed excited states are as follows;

$$\alpha^i_{-1}|0\rangle_{\mathrm{NS-NS'}}, \ \ \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{\mathrm{NS-NS'}} \tag{11.2.10}$$

but because of the GSO projection, A and B should be from the same left subsector. So, the left sector has $SO(8) \times SO(16) \times SO(16)$ symmetry. The α state lives in $(\mathbf{8}_v, \mathbf{1}, \mathbf{1})$ representation of this symmetry and λ states live in either $(\mathbf{1}, \mathbf{120}, \mathbf{1})$ representation or $(\mathbf{1}, \mathbf{1}, \mathbf{120})$ representation. $\mathbf{120}$ comes from the fact that $SO(16)$ has an adjoint representation of dimension $16 \times 15/2 = 120$.
For the $\mathrm{R-NS'}$ sector, we have 16 periodic and 16 anti-periodic fermions and thus, the normal ordering constant is;

$$-\frac{8}{24} + \frac{16}{24} - \frac{16}{48} = 0$$

So, there is no tachyon in the left sector now. Since there is a R sector now, we have 16 zero modes i.e. λ^A_0 for $A = 1, ..., 16$. So, the ground states are degenerate and they live in a $2^{16/2} = 2^8 = 256$ dimensional spinor representation of the first $SO(16)$. The GSO projection retains only one chirality and thus, only $\mathbf{128}$ is retained from $\mathbf{256} = \mathbf{128} + \mathbf{128}'$. These ground states thus live in $(\mathbf{1}, \mathbf{128}, \mathbf{1})$ representation. The same thing happens with $\mathrm{NS-R}$ sector but with $SO(16)'$ instead of $SO(16)$. Thus, the ground state in this sector lives in $(\mathbf{1}, \mathbf{1}, \mathbf{128})$ representation. Notice that $\mathbf{120}$ is the adjoint representation and $\mathbf{128}$ is a spinor representation of $SO(16)$. For the $\mathrm{R-R'}$ sector, the normal ordering sector is again 1 and thus, there are no massless states in the $\mathrm{R-R'}$ spectrum.
The left moving massless spectrum is thus written as follows;

$$(\mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{120} + \mathbf{128}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{120} + \mathbf{128})$$

Coupling this left spectrum with the right spectrum of $\mathbf{8}_v \times \mathbf{8}$, we get the following;

$$(\mathbf{8}_v \times \mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{120} + \mathbf{128}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{1}, \mathbf{120} + \mathbf{128}) + (\mathbf{8} \times \mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{8}, \mathbf{120} + \mathbf{128}, \mathbf{1}) + (\mathbf{8}, \mathbf{1}, \mathbf{120} + \mathbf{128})$$

$$= (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}, \mathbf{1})$$

$$+ (\mathbf{8}_v, \mathbf{120} + \mathbf{128}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{1}, \mathbf{120} + \mathbf{128}) + (\mathbf{8}, \mathbf{120} + \mathbf{128}, \mathbf{1}) + (\mathbf{8}, \mathbf{1}, \mathbf{120} + \mathbf{128})$$

Now, if we want the massless vectors to be in the adjoint representation (**why do we need it?, I know a reason but is there a better reason? Spacetime consistency?**) then we need to find a group with $120 + 128 = 248$ generators such that it has $SO(16)$ as a subgroup and under $SO(16)$, it transforms like **120+128** of $SO(16)$. E_8 is such a subgroup (**Find more about its justification**). So, the full symmetry in the left sector is $SO(8) \times E_8 \times E_8$. Thus, the full massless spectrum is as follows;

$$(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{28},\mathbf{1},\mathbf{1}) + (\mathbf{35},\mathbf{1},\mathbf{1}) + (\mathbf{8}',\mathbf{1},\mathbf{1}) + (\mathbf{56},\mathbf{1},\mathbf{1})$$

$$+(\mathbf{8}_v,\mathbf{248},\mathbf{1}) + (\mathbf{8}_v,\mathbf{1},\mathbf{248}) + (\mathbf{8},\mathbf{248},\mathbf{1}) + (\mathbf{8},\mathbf{1},\mathbf{248})$$

(Talk about bosonisation currents and modular invariance.)

11.3 Other ten-dimensional heterotic strings

11.4 A little lie algebra

Let the Lie algebra G generators be T^a. The Lie bracket of the corresponding Lie algebra $\mathfrak{g}$ is;

$$\left[T^a, T^b\right] = i f^{ab}_{\ \ c} T^c \tag{11.4.1}$$

which satisfies the Jacobi identity;

$$[T^a,[T^b,T^c]] + [T^c,[T^a,T^b]] + [T^b,[T^c,T^a]] = 0 \tag{11.4.2}$$

The finite Lie group element is as follows;

$$\mathbb{I} + i\theta_a T^a + \frac{1}{2!}(\theta_a T^a)^2 + ... = \exp\left(i\theta_a T^a\right) \tag{11.4.3}$$

For a compact group (**why only for this?**), the Lie algebra has an inner product (called the Cartan-Killing form). We define it by defining it for the generators as follows;

$$d^{ab} = \left(T^a, T^b\right) = \mathrm{Tr}\left(\mathrm{ad}(T^a)\mathrm{ad}(T^b)\right) \tag{11.4.4}$$

where $\mathrm{ad}(T^a)$ means T^a in the adjoint representation. This product is symmetric due to the cyclic property of the trace;

$$(T^a, T^b) = \mathrm{Tr}\left(\mathrm{ad}(T^a)\mathrm{ad}(T^b)\right) = \mathrm{Tr}\left(\mathrm{ad}(T^b)\mathrm{ad}(T^a)\right) = (T^b, T^a) \tag{11.4.5}$$

This inner product has the following property as well;

$$\left([T^a,T^b],T^c\right) + \left(T^b,[T^a,T^c]\right) = \mathrm{Tr}\left(T^aT^bT^c\right) - \mathrm{Tr}\left(T^bT^aT^c\right) + \mathrm{Tr}\left(T^bT^aT^c\right) - \mathrm{Tr}\left(T^bT^cT^a\right) = 0 \tag{11.4.6}$$

where we will have to use the cyclic property on the last trace to cancel it with the first trace. This condition is equivalent to the following;

$$\left([T^a,T^b],T^c\right) + \left(T^b,[T^a,T^c]\right) = i f^{ab}_{\ \ d}(T^d,T^c) + i f^{ac}_{\ \ d}(T^b,T^d) = 0 \Rightarrow f^{ab}_{\ \ d} d^{dc} + f^{ac}_{\ \ d} d^{bd} = 0 \tag{11.4.7}$$

A **subalgebra** $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a subset of $\mathfrak{g}$ such that if $X, Y \in \mathfrak{h}$, then $[X,Y] \in \mathfrak{h}$. An **ideal** I of $\mathfrak{g}$ is a subalgebra such if $X \in \mathfrak{g}$ and $Y \in I$, then $[X,Y] \in I$.
Now, $\mathfrak{g}$ has no non-trivial ideals, then $\mathfrak{g}$ is called a **simple** Lie algebra. A Lie algebra is $\mathfrak{g}$ is called **semi-simple** if it has no abelian ideals. We can establish a theorem (**prove it if possible**) that a semi-simple Lie algebra is a direct sum of simple Lie algebras. For simple Lie algebra, the inner product is unique (**prove this**). It can be set to δ^{ab} by choosing a basis. We can generalize (11.4.4) to arbitrary representations as follows;

$$\mathrm{Tr}\left(t_r^a t_r^b\right) = T_r d^{ab}$$

where t_r^a is the generator in r representation of $\mathfrak{g}$ and T_r is an r dependent constant. We see from (11.4.4) that adjoint representation, $T_r = 1$.
We also see that the following is true;

$$\left[t_r^a t_r^b d_{ab}, t_r^c\right] = d_{ab}\left[t_r^a, t_r^c\right] t_r^b + d_{ab} t_r^a \left[t_r^b, t_r^c\right] = i d_{ab} f^{ac}_{\ \ d} t_r^d t_r^b + i d_{ab} f^{bc}_{\ \ d} t_r^a t_r^d$$

Renaming the indices in the second term as $a \to d,\ d \to b,\ b \to a$, we get;

$$\left[t_r^a t_r^b d_{ab}, t_r^c\right] = -i\left(d_{da} f^{ca}{}_b + d_{ab} f^{ca}{}_d\right) t_r^d t_r^b = -i\left(f^c{}_{db} + f^c{}_{bd}\right) t_r^d t_r^b = 0 \tag{11.4.8}$$

where we used the antisymmetry of $f^a{}_{bc}$ in the last step. So, $t_r^a t_r^b d_{ab}$ commutes with all generators, and thus, (by Schur's lemma), it should be a constant but dependent on r. So, we have;

$$t_r^a t_r^b d_{ab} = Q_r$$

where Q_r is a constant called the **Casimir invariant of** r. The well-known Lie groups series and their Lie algebras are as follows;

- $A_n = SO(n+1)$ for $n > 1$. The matrices are unitary matrices with determinant 1. The infinitesimal $U \in SO(n)$ manipulation gives us the following;

  $$U_{ab} = \mathbb{I}_{ab} + i\theta_\alpha T^\alpha_{ab} + \mathcal{O}(\theta^2) \Rightarrow U^{-1}_{ab} = U^\dagger_{ab} = \mathbb{I}_{ab} - i\theta_\alpha (T^{*\alpha})^T_{ab} + \mathcal{O}(\theta^2)$$
  $$\Rightarrow \mathbb{I}_{ab} = (UU^{-1})_{ab} = \mathbb{I} + i\theta_\alpha\left(T^\alpha_{ab} - T^{*\alpha}_{ba}\right) + \mathcal{O}(\theta^2) \Rightarrow T^\alpha_{ab} = T^{*\alpha}_{ba}$$

  Moreover, we have the following;

  $$1 = \det U = \det\left(\mathbb{I} + i\theta_\alpha T^\alpha + \mathcal{O}(\theta^2)\right) = 1 + i\theta_\alpha \operatorname{Tr}(T^\alpha) + \mathcal{O}(\theta^2) \Rightarrow \operatorname{Tr}(T^\alpha) = 0$$

  So, the generators are traceless Hermitian matrices.

- $B_n = SO(2n+1),\ D_n = SO(2n)$ for $n > 1$. The infinitesimal $M \in SO(2n)$ or $M \in SO(2n+1)$ is given as follows;

  $$M_{ab} = \mathbb{I}_{ab} + \theta_\alpha T^\alpha_{ab} + \mathcal{O}(\theta^2) \Rightarrow M^{-1}_{ab} = M^T_{ab} = \mathbb{I}_{ab} + \theta_\alpha (T^\alpha)^T_{ab} + \mathcal{O}(\theta^2)$$
  $$\Rightarrow \mathbb{I}_{ab} = (MM^T)_{ab} = \mathbb{I} + \theta_\alpha\left(T^\alpha_{ab} + (T^\alpha)^T_{ba}\right) + \mathcal{O}(\theta^2) \Rightarrow T^\alpha_{ab} + T^{T\alpha}_{ba} = 0$$

  So, T is antisymmetric and thus, also traceless. Moreover, we have the following;

  $$1 = \det M = \det\left(\mathbb{I} + \theta_\alpha T^\alpha + \mathcal{O}(\theta^2)\right) = 1 + \theta_\alpha \operatorname{Tr}(T^\alpha) + \mathcal{O}(\theta^2) \Rightarrow \operatorname{Tr}(T^\alpha) = 0$$

  But this gives us no new information. So, the generators are antisymmetric matrices.

- $C_n = USp(k)$ which consists of $2n \times 2n$ unitary matrices such that if $U \in USp(n)$, then;

  $$MUM^{-1} = (U^T)^{-1} \quad \text{where } M = i\begin{bmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{bmatrix} \Rightarrow M^{-1} = M$$

  The infinitesimal version of U is as follows;

  $$U = \mathbb{I} + i\theta_\alpha T^\alpha + \mathcal{O}(\theta^2) \Rightarrow MUM^{-1} = \mathbb{I} + i\theta_\alpha M T^\alpha M^{-1} + \mathcal{O}(\theta^2) \tag{11.4.9}$$

  Since U is unitary, T^α's should be hermitian. Moreover, we have;

  $$U^T = \mathbb{I} + i\theta_\alpha (T^\alpha)^T + \mathcal{O}(\theta^2) \Rightarrow (U^T)^{-1} = \mathbb{I} - i\theta_\alpha (T^\alpha)^T + \mathcal{O}(\theta^2) \tag{11.4.10}$$

  Comparing (11.4.9) and (11.4.10), we get;

  $$MT^\alpha M^{-1} = -(T^\alpha)^T \Rightarrow MTM^{-1} = -T^T\ \forall \alpha \tag{11.4.11}$$

  So, the generators satisfy the condition in (11.4.11). We now constrain the form of T by this information. First of all, we write T in the block form (where all the matrices are $n \times n$ matrices);

  $$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

  Now, using (11.4.11), we have;

  $$-\begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix} = -\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \Rightarrow \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} -A^T & -C^T \\ -B^T & -D^T \end{pmatrix}$$

  This implies;

  $$D = -A^T,\ C = C^T,\ B = B^T \Rightarrow T = \begin{pmatrix} A & C \\ B & -A^T \end{pmatrix} \tag{11.4.12}$$

with B and C being symmetric matrices. Now, we use the fact that T is hermitian. Then, we have;

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow A = A^\dagger, B = C^\dagger, C = B^\dagger, D = D^\dagger \tag{11.4.13}$$

Therefore, using (11.4.13) and (11.4.12) we get;

$$D = D^\dagger = (-A^T)^\dagger = -A^*$$

$$B = C^\dagger = (C^T)^* = C^* \Rightarrow C = B^*$$

Therefore, we have;

$$T = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix} \quad \text{with } A^\dagger = A,\ B^T = B \tag{11.4.14}$$

(derive the number of independent components and about the non-compact symplectic group). We divide the set of generators into two sets. Let H be the set of all the generators that commute among themselves but H is the maximal such set (the elements of this set are called H^i). The number of elements in H is called the rank of . The subspace of that is spanned by the elements of H is called the **Cartan subalgebra**. The other set E contains the rest of the generators. The Lie algebra can then be written as follows;

$$[H^i, H^j] = 0,\ \ [H^i, E^\alpha] = \alpha^i E^\alpha \tag{11.4.15}$$

We will first show that H^j's can't appear in $[E^\alpha, E^\beta]$ commutator if $\alpha + \beta \neq 0$. The most general form of this commutator is as follows;

$$[E^\alpha, E^\beta] = f_i^{\alpha\beta} H^i + \zeta_\mu^{\alpha\beta} E^\mu \tag{11.4.16}$$

with not all f's vanishing. Taking commutator with H^j gives us the following;

$$[H^j, [E^\alpha, E^\beta]] = \zeta_\mu^{\alpha\beta}[H^j, E^\mu] = \zeta_\mu^{\alpha\beta}\mu^j E^\mu \tag{11.4.17}$$

We calculate this commutator using Jacobi identity and (11.4.15) as follows;

$$[H^j, [E^\alpha, E^\beta]] + [E^\beta, [H^j, E^\alpha]] + [E^\alpha, [E^\beta, H^j]] = 0$$

$$\Rightarrow [H^j, [E^\alpha, E^\beta]] = [[H^j, E^\alpha], E^\beta] + [E^\alpha[H^j, E^\beta]] = (\alpha^j + \beta^j)[E^\alpha, E^\beta] = (\alpha^j + \beta^j)\left(f_i^{\alpha\beta} H^i + \zeta_\mu^{\alpha\beta} E^\mu\right) \tag{11.4.18}$$

Comparing (11.4.18) with (11.4.17), we see that if $\alpha + \beta \neq 0$, then all $f_i^{\alpha\beta}$ are zero. So, we have proved what we wanted to show. We also notice that;

$$\zeta_\mu^{\alpha\beta}(\alpha^j + \beta^j - \mu^j) = 0\ \ \forall \alpha, \beta$$

Therefore, we see that $\zeta_\mu^{\alpha\beta} \neq 0$ only when $\alpha + \beta = \mu$ where μ is a root. If $\alpha + \beta$ is a root, then $\zeta_\mu^{\alpha\beta} = 0$ for all μ except $\mu = \alpha + \beta$. We denote $\zeta_{\alpha+\beta}^{\alpha\beta}$ as $\epsilon(\alpha, \beta)$.

We might still have H in (11.4.16) if $\alpha + \beta = 0$ and thus, we rewrite $f_i^{\alpha\beta}$ as f_i^α only because $f_i^{\alpha\beta}$ is non-zero only if $\beta = -\alpha$. We see by comparing (11.4.17) and (11.4.18) that if $\alpha + \beta = 0$, then $\zeta_\mu^{\alpha\beta}\mu^j = 0$ for all μ. But all μ can't be zero in general because that will make the whole lie algebra equal to Cartan subalgebra but we know that this isn't true in general. So, for $\beta = -\alpha$, all $\zeta_\mu^{\alpha\beta}$'s vanish. Therefore, we see that;

$$[E^\alpha, E^\beta] = \epsilon(\alpha, \beta) E^{\alpha+\beta}\ \ (\alpha + \beta \text{ is a root}),\ \ [E^\alpha, E^\beta] = f_i^\alpha H^i\ \ (\alpha + \beta = 0),\ [E^\alpha, E^\beta] = 0 \text{ (Otherwise)}$$

It is easy to see that $\epsilon(\beta, \alpha) = -\epsilon(\alpha, \beta)$. The possible values of $\epsilon(\alpha, \beta)$ are ± 1 (**derive this**). Using a particular (EEE) Jacobi identity, we have (with $\beta \neq \alpha$ and $\beta \neq -\alpha$);

$$[E^\alpha, [E^{-\alpha}, E^\beta]] + [E^\beta, [E^\alpha, E^{-\alpha}]] + [E^{-\alpha}, [E^{-\beta}, E^\alpha]] = 0 \Rightarrow f_i^\alpha \beta^i = 0$$

Similarly, for $\beta = \alpha$, we get $f_i^\alpha \alpha^i = 0$. At the end, we get (**derive this**);

$$f_i^\alpha = \frac{2\alpha_i}{\alpha^2} \Rightarrow [E^\alpha, E^{-\alpha}] = \frac{2\alpha_i}{\alpha^2} H^i = \frac{2\alpha.H}{\alpha^2} \tag{11.4.19}$$

The matrices that represent H^i's are mutually commuting and thus, they have simultaneous eigenvectors. Let the eigenvalues of the $j-$th eigenvector under the matrix representing H^i be w^i_j. We can get weight vectors where the $k-$th weight vector is;

$$(w_k^1, w_k^2, ..., w_k^{\text{rank}(\mathfrak{g})}), \quad k = 1, 2, 3, ..., \dim(r)$$

The number of weight vectors is equal to the dimension of the representation. For reference, we quote the $\alpha-$th root vector as well;

$$\left(\alpha^1, \alpha^2, ..., \alpha^{\text{rank}}(\mathfrak{g})\right), \quad \alpha = 1, ..., \dim \mathfrak{g}$$

$\dim \mathfrak{g}$ equals $\dim(r)$ when r is the adjoint representation. So, it turns out that the weight vectors of adjoint representation are the same as the root vectors (**derive this**). Examples of the weights and roots are as derived now.

- The vector representation of the Cartan subalgebra for $SO(2n)$ consists of the matrices that have all entries equal to zero but a 2×2 submatrix. H^j has the following non-zero 2×2 submatrix in $(2j-1, 2j)$ rows and columns;

  $$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

  Therefore, we have;

  $$H^j_{ab} = i\left(\delta_{a+1-2j}\delta_{b-2j} - \delta_{a-2j}\delta_{b+1-2j}\right), \quad a, b = 1, ..., 2n$$

  (Show that they commute). The rank of $SO(2n)$ is thus n. The eigenvectors of H^j are such that they have zeros except in the $(2j-1, 2j)$ positions. The eigenvectors are as follows

  $$V^j_\pm = (0, ..., 0, 1, \mp i, ...0)$$

  We just manipulate the 2 dimensional spaces to find the eigenvalues;

  $$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \mp i \end{bmatrix} = \begin{bmatrix} \pm 1 \\ -i \end{bmatrix} = \pm \begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

  So, the eigenvalues of $V^j_\pm$ under H^k are as follows;

  $$\omega^j_\pm = (0, ..., \pm1, ..., 0)$$

  where the nonzero entry is in the $j-$th position. So, $\omega^j_\pm$ are the weight vectors of the vector representation.

  The adjoint representation is the product of two vector representations (**justify this**) and the weights add under the products (**justify this**). So, there are four possibilities. We can add a ω^j_+ with ω^k_+ which gives us $+1$ in $j-$th and $k-$th positions. Moreover, we can add a ω^j_- with ω^k_- which gives us -1 in $j-$th and $k-$th positions. In all of these possibilities, $j \neq k$ because of the antisymmetry of the tensor representation (**justify**). However, we add a ω^j_+ with ω^k_- which gives us a $+1$ in $j-$th position and -1 in $k-$th position for $j \neq k$. However, if $j = k$, then we get a zero root. There are k such roots. Since these are zero roots, they obviously correspond to H^i for $(i = 1, ..., n)$. Therefore, the weight vectors for the adjoint representation which are also the root vectors are as follows;

  $$(+1, +1, \underbrace{0, ..., 0}_{n-2}) + \text{permutations} \tag{11.4.20}$$

  $$(+1, -1, \underbrace{0, ..., 0}_{n-2}) + \text{permutations} \tag{11.4.21}$$

  $$(-1, -1, \underbrace{0, ..., 0}_{n-2}) + \text{permutations} \tag{11.4.22}$$

  $$(\underbrace{0, ..., 0}_{n}) \text{ (n of them)} \tag{11.4.23}$$

  The weights for spinor representations are as follows (**justify**). The entries in the weight vectors are $\pm1/2$. For 2^{n-1} representation, the weight vectors have an even number of $1/2$ and for 2^{n-1} representation, the weight vectors have an odd number of $1/2$.

- $B_n = SO(2n+1)$ has the same Cartan generators with an additional row and column of zeros. So, the weight vectors are the same as $SO(2n)$ case but we also have another eigenvector for H^i's and it is just the zeros. The eigenvalues for this eigenvector for all H^i's are obviously zeros. So, there is an additional weight vector i.e.;

  $$\underbrace{(0,...,0)}_{n}$$

  The number of generators of $SO(2n)$ and $SO(2n+1)$ are as follows;

  $$SO(2n): \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2+n, \quad SO(2n+1): \frac{(2n+1)(2n+2)}{2} = (n+1)(2n+1) = 2n^2+3n+1$$

  So, we have an additional $2n^2+3n+1-2n^2-n = 2n+1$ generators in $SO(2n+1)$ as compared to $SO(2n)$. The root vectors corresponding to these additional $2n+1$ generators are as follows (**justify**);

  $$(\pm 1, \underbrace{0,...,0}_{n-1}) + \text{permutations}$$

  (**One root vector missing. Do the other algebras**).

We also define the **dual Coexter number** $h(g)$ as follows;

$$-\sum_{b,c} f^{ab}_{\ \ c} f^{dc}_{\ \ b} = h(g)\psi^2 d^{ad} \tag{11.4.24}$$

where $\psi^2 = \psi^i\psi^j d_{ij}$. This means that $\psi^2 d^{ad} = \psi^i\psi^j d_{ij} d^{ad}$ and thus, the normalization of ψ^2 doesn't change $\psi^2 d^{ad}$ and hence, the normalization also doesn't change $h(g)$. The properties of the Lie algebras are summarized as follows (**derive these**).

Group	**Simply-laced**	**dimension**	$h(g)$
$A_{n-1} = SU(n)(n>1)$	Yes	n^2-1	n
$B_n = SO(2n+1)(n\geq 1)$	No	$2n(n+1)$	$2n-1$
$C_n = USp(n)(n\geq 1)$	No	$2n^2+n$	$n+1$
$D_n = SO(2n)(n\geq 1)$	Yes	$n(2n+1)$	$2n-2$
E_6	Yes	78	12
E_7	Yes	133	18
E_8	Yes	248	30
F_4	?	52	9
G_2	?	14	4

11.4.1 Other useful facts for grand unification

tt

11.5 Current algebras

Let $j^a(z)$ be a field with conformal weights $(1,0)$ (also called a holomorphic current). The OPE of j^a with j^b is constrained by conformal invariance (prove this) and it is as follows;

$$j^a(z)j^b(0) \sim \frac{k^{ab}}{z^2} + \frac{ic^{ab}_c}{z} j^c(0)$$

where k^{ab} and c^{ab}_c are constants. Note that both terms have the same conformal weight as they should. The mode expansion of $j^a(z)$ is as follows;

$$j^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j^a_n \Rightarrow j^a_n = \oint_{C(0)} \frac{dz}{2\pi i} z^n j^a(z)$$

where $C(0)$ is the contour around 0. The commutation relation can be derived via normal procedure as follows;

$$\left[j^a_n, j^b_m\right] = \oint_{C(0)} \frac{dz}{2\pi i} \oint_{C(0)} \frac{dw}{2\pi i} w^m z^n j^a(z) j^b(w) = \oint_{C(0)} \frac{dw}{2\pi i} w^m \oint_{C(w)} \frac{dz}{2\pi i} z^n \mathcal{R}\left(j^a(z)j^b(w)\right)$$

$$= \oint_{C(0)} \frac{dw}{2\pi i} w^m \oint_{C(w)} \frac{dz}{2\pi i} z^n \left(\frac{k^{ab}}{(z-w)^2} + \frac{ic_c^{ab}}{z-w} j^c(0) \right) = \oint_{C(0)} \frac{dw}{2\pi i} w^m \left(k^{ab} n w^{n-1} + ic_c^{ab} j^c(w) w^n \right)$$

$$= nk^{ab} \oint_{C(0)} \frac{dw}{2\pi i} w^{m+n-1} + ic_c^{ab} \oint_{C(0)} \frac{dw}{2\pi i} j^a(w) w^{m+n} = ic_c^{ab} j^a_{m+n} + nk^{ab} \delta_{m+n} \tag{11.5.1}$$

where $\mathcal{R}$ denotes the radial ordering operator which just picks the singular OPE of $j^a j^b$ (**write more about it**). (11.5.1) also implies;

$$[j_0^a, j_0^b] = ic_c^{ab} j_0^a \tag{11.5.2}$$

So, j_0^a's span a Lie algebra with $f_c^{ab} = c_c^{ab}$. Let's call this Lie algebra g. This means that the following is satisfied;

$$c_d^{bc} c_e^{ad} + c_d^{ab} c_e^{cd} + c_d^{ca} c_e^{bd} = 0 \tag{11.5.3}$$

The $j_1^a j_0^b j_{-1}^c$ Jacobi identity gives us the following;

$$[j_1^a, [j_0^b, j_{-1}^c]] + [j_{-1}^c, [j_1^a, j_0^b]] + [j_0^b, [j_{-1}^c, j_1^a]] = 0$$

$$\Rightarrow \left(c_d^{bc} c_e^{ad} + c_d^{ab} c_e^{cd} + c_d^{ca} c_e^{bd} \right) j_0^e + k^{ad} c_d^{bc} + k^{cd} c_d^{ab} = 0 \Rightarrow k^{ad} c_d^{bc} + k^{cd} c_d^{ab} = 0$$

where we used (11.5.3) in the last step. Using (11.4.7) we see that if g is simple, then k^{ab} is proportional to d^{ab} since the inner product is unique for simple g. So, we have;

$$k^{ab} = \hat{k} d^{ab}$$

where $\hat{k}$ is a constant. We also see that $\hat{k}$ is positive if the theory is unitary. If $|1\rangle$ is the state corresponding to the unit operator, then we have;

$$k^{ab} = \hat{k} d^{aa} = [j_1^a, j_{-1}^a] = [j_1^a, j_{-1}^a]\langle 1|1\rangle = \langle 1|[j_1^a, j_{-1}^a]|1\rangle = \langle 1|j_1^a, j_{-1}^a|1\rangle - \langle 1|j_{-1}^a, j_1^a|1\rangle = \langle 1|j_1^a, j_{-1}^a|1\rangle = ||j_{-1}^a|1\rangle||^2 \geq 0$$

where we used the fact that j_1 should annihilate $|1\rangle$. Now, for compact Lie algebra, d^{aa} is positive definite (**prove this**) and thus, $\hat{k} \geq 0$. Since the weight of j^a is 1, the vertex operator of j_{-1}^a is $j^a(z)$ and thus, $\hat{k}$ vanishes only if $j^a(z)$ vanishes.

To show that $\hat{k}$ is quantized, take any root of g. Denote this root by α. Define three generators as follows;

$$J^3 = \frac{\alpha.H}{\alpha^2}, \ J^\pm = E^{\pm\alpha}$$

Then, we can derive that;

$$[J^3, J^\pm] = \left[\frac{\alpha.H}{\alpha^2}, E^{\pm\alpha} \right] = d_{ij} \frac{\alpha^i}{\alpha^2} [H^j, E^{\pm\alpha}] = \pm d_{ij} \frac{\alpha^i}{\alpha^2} [H^j E^{\pm\alpha}] = \pm E^{\pm\alpha} = \pm J^\pm$$

$$[J^+, J^-] = \frac{2\alpha.H}{\alpha^2} = 2J^3$$

From the $SU(2)$ representation theory, we know that the representations of $SU(2)$ are labeled by the eigenvalue of J^3 and they are integers or half integers. So, $2J^3$ is always an integer. So, $2\alpha.H/\alpha^2$ has integer eigenvalues.

There is another $SU(2)$ subalgebra in the current algebra. For that, let's call the Cartan generators as H_0 and define the following;

$$J^3 = \frac{\alpha.H_0 + \hat{k}}{\alpha^2}, \quad J^\pm = E_1^{\pm\alpha}$$

The 1 subscript on E denotes that it is made by linear combinations of j_1^a and j_{-1}^a. (**Prove that they also satisfy the SU(2) algebra by writing the exact forms of $E_1^{\pm\alpha}$**). So, we deduce that $2\hat{k}/\alpha^2$ is also an integer for all roots. If this condition is satisfied for long roots, then it is automatically satisfied for short roots. So, we have;

$$k = \frac{2\hat{k}}{\psi^2} \in \mathbb{Z} \tag{11.5.4}$$

where ψ is a long root. k is called the level. **(Write about different normalizations)**. From now, we will take $d^{ab} = \delta^{ab}$. For $\mathfrak{g} = U(1) \times U(1) \times ... \times U(1)$, there are no roots (because the generators commute) and thus, there is no concept of a level. Moreover, $c^{ab}_c = 0$ because the group is abelian (so, there is no z^{-1} term in the jj OPE). So, we can normalize the OPE to the following;

$$j^a(z)j^b(0) \sim \frac{\delta^{ab}}{z^2}$$

This OPE matches the OPE of the free boson current OPE which is as follows;

$$i\partial H^a(z) i\partial H^b(0) \sim \frac{\delta^{ab}}{z^2}$$

So, we have the equivalence $j^a \sim i\partial H^a$. (**check the equivalence between stress tensors as well**). An example of currents is the fermion bilinear;

$$i\lambda^A\lambda^B$$

which lives in the vector × vector representation of $SO(n)$. The z^{-2} term of the OPE of this current is as follows;

$$i\lambda^A\lambda^B(z) i\lambda^A\lambda^B(w) = -\lambda^A\lambda^B(z)\lambda^A\lambda^B(w) \sim -\lambda^A\lambda^B(z)\lambda^A\lambda^B(w) = \frac{1}{(z-w)^2}$$

where we used the anti-commutativity of fermions and the OPE of λ^A which is as follows;

$$\lambda^A(z)\lambda^B(w) \sim \frac{\delta^{AB}}{z-w}$$

So, the coefficient of $1/(z-w)^2$ term is 1. So, for this current, we compare this to the result of the general currents;

$$1 = \hat{k}d^{aa} = \hat{k}\delta^{aa} = \psi^2 k/2$$

d^{aa} appears because we took the OPE of a current with itself (i.e. $i\lambda^A\lambda^B$ with itself). Now, the long roots of $SO(n)$ have length squared equal to 2 (except $n = 3$). So, we have;

$$k = 1 \text{ for } i\lambda^A\lambda^B \ (n \neq 3)$$

For $n = 3$, there is no long root and $\psi^2 = 1$ and thus, $k/2 = 1 \Rightarrow k = 2$. **(Why rescale diagonal current?)**. Another example can be constructed from any real representation r. Take $\dim(r)$ real fermions (**Why everything real?**) and construct the following current;

$$\frac{1}{2}\lambda^A\lambda^B t_{r,AB}$$

The z^{-2} term in the OPE is as follows;

$$\frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w) \sim \frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w) + \frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w)$$

$$= \frac{1}{4(z-w)^2}t_{r,AB}t_{r,CD}\left(-\delta^{AC}\delta^{BD} + \delta^{BC}\delta^{AD}\right) = \frac{1}{4(z-w)^2}\left(t_{r,AB}t_{r,BA} - t_{r,AB}t_{r,AB}\right) = \frac{1}{4(z-w)^2}\left(\mathrm{Tr}(t_r t_r) - \mathrm{Tr}(t_r t_r^T)\right)$$

Now, using the fact that $\mathrm{Tr}(t_r t_r) = -\mathrm{Tr}(t_r t_r^T) = T_r$ (**justify the second equality**), we get;

$$\frac{1}{4}t_{r,AB}t_{r,CD}\lambda^A(z)\lambda^B(z)\lambda^C(w)\lambda^D(w) \sim \frac{T_r}{2(z-w)^2}\left(+\mathcal{O}(z^{-1})\right)$$

Comparing it to the general case, we have;

$$T_r/2 = k^{aa} = \hat{k}d^{aa} = \hat{k}\delta^{aa} = \hat{k} = \psi^2 k/2 \Rightarrow T_r = \psi^2 k \Rightarrow k = T_r/\psi^2$$

(Write the other examples)

11.5.1 Sugawara construction

We now show that $:jj:(z)$ has the same OPE with $j^c(w)$ as the stress tensor $T_B(z)$ which is as follows;

$$T_B(z)j^c(w) \sim \frac{1}{(z-w)^2}j^c(w) + \frac{1}{z-w}\partial j^c(w) \tag{11.5.5}$$

since $j^c(w)$ is a Virasoro primary field and has conformal weight $(1,0)$. Now, $:jj:(z)$ is defined as follows'

$$:jj:(z) = \lim_{z_1\to z}\left(j^a(z)j^a(z_1) - \frac{\hat{k}\,\delta^{aa}}{(z-z_1)^2}\right) = \lim_{z_1\to z}\left(j^a(z)j^a(z_1) - \frac{\hat{k}\,\dim(g)}{(z-z_1)^2}\right)$$

where the sum over a is implicit. We consider the following OPE such that z_1 and z_2 are far;

$$\begin{aligned}
j^a(z_1)j^a(z_2)j^c(z_3) &\sim \overbrace{j^a(z_1)j^a(z_2)j^c}(z_3) + j^a(z_1)\overbrace{j^a(z_2)j^c}(z_3)\\
&= \frac{\hat{k}j^c(z_2)}{(z_1-z_3)^2} + \frac{if^{acd}}{z_1-z_3}j^a(z_2)j^d(z_3) + \frac{\hat{k}j^c(z_1)}{(z_2-z_3)^2} + \frac{if^{acd}}{z_2-z_3}j^a(z_1)j^d(z_3)\\
&= \frac{\hat{k}j^c(z_2)}{(z_1-z_3)^2} + \frac{if^{acd}}{z_1-z_3}j^a(z_2)j^d(z_1) + \frac{\hat{k}j^c(z_1)}{(z_2-z_3)^2} + \frac{if^{acd}}{z_2-z_3}j^a(z_1)j^d(z_2) + (\text{holomorphic terms in } z_3)
\end{aligned}$$

Now, we expand this expression as a Laurent expansion in (z_2-z_1). We keep terms up to $(z_2-z_1)^0$ only. It is as follows;

$$\frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}+\frac{if^{acd}}{z_1-z_3}\left[\frac{\hat{k}\delta^{ad}}{(z_2-z_1)^2}+\frac{if^{ade}j^e(z_1)}{z_2-z_1}\right]+\frac{\hat{k}j^c(z_1)}{(z_2-z_1+z_1-z_3)^2}+\frac{if^{acd}}{z_2-z_1+z_1-z_3}\left[\frac{\hat{k}\delta^{ad}}{(z_1-z_2)^2}+\frac{if^{ade}j^e(z_2)}{z_1-z_2}\right]$$

$$\begin{aligned}
&\sim \frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2} - \frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)} + \frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2}\left[1+\frac{(z_2-z_1)}{(z_1-z_3)}\right]^{-2} + \frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)}\left[1+\frac{(z_2-z_1)}{(z_1-z_3)}\right]^{-1}\\
&\sim \frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2} - \frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)} + \frac{\hat{k}j^c(z_1)}{(z_1-z_3)^2} + \frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)(z_2-z_1)} - \frac{f^{acd}f^{ade}j^e(z_1)}{(z_1-z_3)^2}\\
&\sim \frac{2\hat{k}j^c(z_1)}{(z_1-z_3)^2} + \frac{f^{cad}f^{ead}j^e(z_1)}{(z_1-z_3)^2}
\end{aligned}$$

where we made heavy use of antisymmetry of f^{abc} and dropped all terms of the order of $\mathcal{O}(z_2-z_1)$ or higher. Using (11.4.24) (where we have kept all the indices upwards), we have;

$$f^{cad}f^{ead} = \psi^2 h(g)\delta^{ce}$$

Therefore, we have (**what about the other term in definition? See that**);

$$:jj:(z_1)j^c(z_3) \sim \frac{2\hat{k}+\psi^2h(g)}{(z_1-z_3)^2}j^c(z_1) \sim (k+h(g))\,\psi^2\left(\frac{1}{(z_1-z_3)^2}j^c(z_3) + \frac{1}{z_1-z_3}\partial j^c(z_3)\right)$$

where we used the fact $2\hat{k} = \psi^2 k$. Therefore, we see that;

$$\frac{:jj:(z_1)}{(k+h(g))\,\psi^2}j^c(z_3) \sim \frac{1}{(z_1-z_3)^2}j^c(z_3) + \frac{1}{z_1-z_3}\partial j^c(z_3)$$

This OPE matches (11.5.5) and we can thus define;

$$T^s_B(z) = \frac{:jj:(z)}{(k+h(g))\,\psi^2} \tag{11.5.6}$$

where s stands for Sugawara. We can also derive the following result;

$$T^s_B(z_1)j^c(z_3) \sim \frac{1}{(z_1-z_3)^2}j^c(z_3) + \frac{1}{z_1-z_3}\partial j^c(z_3)$$

$$\Rightarrow j^c(z_3)T_B^s(z_1) \sim \frac{1}{(z_3-z_1)^2}j^c(z_1) + \frac{1}{z_3-z_1}\partial j^c(z_1) - \frac{1}{z_3-z_1}\partial j^c(z_1) = \frac{1}{(z_3-z_1)^2}j^c(z_1)$$

Now, we derive the following OPE where again z_2 and z_1 are far;

$$j^a(z_1)j^a(z_2)T_B^s(z_3) \sim j^a(z_1)j^a(z_2)T_B^s(z_3) + j^a(z_1)j^a(z_2)T_B^s(z_3)$$

$$= \frac{1}{(z_3-z_1)^2}j^a(z_1)j^a(z_2) + \frac{1}{z_3-z_1}\partial j^a(z_1)j^a(z_2) + \frac{1}{(z_3-z_2)^2}j^a(z_1)j^a(z_2) + \frac{1}{z_3-z_2}\partial j^a(z_2)j^a(z_1) \quad (11.5.7)$$

Before doing the expansion in $z_2 - z_1$, we need the following OPEs;

$$j^a(z_1)j^a(z_2) \sim \frac{\hat{k}}{(z_1-z_2)^2} \Rightarrow \partial j^a(z_1)j^a(z_2) \sim \frac{-2\hat{k}}{(z_1-z_2)^3},\ \ j^a(z_1)\partial j^a(z_2) \sim \frac{2\hat{k}}{(z_1-z_2)^3}$$

Now, we do the $z_2 - z_1$ Laurent expansion in (11.5.7) and keep terms up to $(z_2-z_1)^0$ only, we get (**complete the TT OPE calculation**);

$$T_B^s(z_1)T_B^s(z_2) \sim \frac{c^{g,k}}{2(z_1-z_3)^4} + \frac{2}{2(z_1-z_3)^2}T_B^s(z_3) + \frac{1}{z_1-z_3}\partial T_B^s(z_3),\ \ c^{g,k} = \frac{k\,\dim(g)}{k+h(g)}$$

(Write about the vanishing of normal ordering constant). Since the concept of the level doesn't make sense for $U(1)$, we can use (11.5.4) and the identification $j = i\partial H$ to write the stress-energy tensor as follows (recall that we use $\alpha' = 2$ for H CFT);

$$T_B^s(z) = -\frac{1}{2}:\partial H\partial H:(z) = \frac{1}{2}:i\partial H i\partial H:(z) = \frac{1}{2}:jj:(z)$$

The total stress tensor T_B is the sum of the Sugawara tensor T_B^s and the tensor that isn't made by currents that we call T_B' and thus, it has non-singular OPE with j^a's. The $T'T'$ OPE is as follows (**derive this**);

$$T_B'(z_1)T_B'(z_2) \sim \frac{c'}{2(z_1-z_2)^4} + \frac{2T_B(z_2)}{(z_1-z_2)^2} + \frac{1}{z_1-z_2}\partial T_B'(z_2),\ \ c' = c - c^{g,k}$$

If the non-current theory is unitary, then;

$$c - c^{g,k} \geq 0 \Rightarrow c \geq c^{g,k}$$

For any simply laced algebra, we have (**prove this**);

$$h(g) = \frac{\dim(g)}{\text{rank}(g)} - 1$$

and therefore, we have;

$$c^{g,k} = k\,\dim(g)\left[k + \frac{\dim(g)}{\text{rank}(g)} - 1\right]^{-1} = \frac{k\,\dim(g)}{\dim(g) + (k-1)\text{rank}(g)}$$

which means that;

$$c^{g,1} = \text{rank}(g)$$

Now, if $g = E_8 \times E_8$ or $g = SO(32)$, then $\text{rank}(g) = 16$ and thus $c = 16$ which is the same as the central charge of a theory of 32 fermions or 16 bosons. For this to happen in detail for the fermionic theory, the Sugawara tensor should be what we derived while deriving the λ part of T_B in heterotic theory i.e.;

$$T_B^s(z) = -\frac{1}{2}:\lambda^A\partial\lambda^A:(z) \quad (11.5.8)$$

To derive this, we consider the heterotic $SO(32)$ current as follows;

$$j^{(A,B)}(z) = i\lambda^A\lambda^B(z),\ \ A,B = 1,...,32,\ A \neq B$$

We see that $A \leftrightarrow B$ gives the same current but with a negative sign. Now, $:j^{(A,B)}j^{(A,B)}:(z)$ is given as follows;

$$:j^{(A,B)}j^{(A,B)}:(z) = \lim_{z\to z_1}\left(\frac{1}{2}\sum_{A\neq B} j^{(A,B)}(z)j^{(A,B)}(z_1) - \frac{\dim(SO(32))}{(z-z_1)^2}\right)$$
$$= \lim_{z\to z_1}\left(\frac{1}{2}\sum_{A,B=1,\ A\neq B}^{32} j^{(A,B)}(z)j^{(A,B)}(z_1) - \frac{496}{(z-z_1)^2}\right)$$

where the factor of $1/2$ comes to avoid counting over equivalent currents. Now, we do the following calculation;

$$j^{(A,B)}(z)j^{(A,B)}(z_1) = -:\lambda^A\lambda^B:(z):\lambda^A\lambda^B:(z_1)$$

$$= -:\lambda^A\lambda^B:(z):\lambda^A\lambda^B:(z_1) - :\lambda^A\lambda^B:(z):\lambda^A\lambda^B:(z_1) - :\lambda^A\lambda^B:(z):\lambda^A\lambda^B:(z_1) + \mathcal{O}(z-z_1)$$
$$= \frac{1}{(z-z_1)^2} + \frac{\lambda^B(z)\lambda^B(z_1)}{z-z_1} + \frac{\lambda^A(z)\lambda^A(z_1)}{z-z_1} + \mathcal{O}(z-z_1)$$
$$= \frac{1}{(z-z_1)^2} - :\lambda^B\partial\lambda^B:(z) - :\lambda^A\partial\lambda^A:(z) + \mathcal{O}(z-z_1)$$

Now, dropping the $\mathcal{O}(z_1-z_2)$ part, we use this result to get;

$$:j^{(A,B)}j^{(A,B)}:(z) = \lim_{z\to z_1}\left[\frac{1}{2}\sum_{A,B=1\ \ A\neq B}^{32}\left(\frac{1}{(z-z_1)^2} - :\lambda^B\partial\lambda^B:(z) - :\lambda^A\partial\lambda^A:(z)\right) - \frac{496}{(z-z_1)^2}\right]$$

The first term in the sum isn't dependent on A or B and thus, we get $32\times 31 = 992$ from the sum but the factor of $1/2$ makes it 496 and thus, it cancels the last term in the limit.
Moreover, the second and third term in the sum gives the same sum but the factor of $1/2$ just reduces it to a single term. Moreover, these terms are independent of z_1 and thus, the limit is trivial. So, we get;

$$:j^{(A,B)}j^{(A,B)}:(z) = -\sum_{A,B=1\ \ A\neq B}^{32} :\lambda^A\partial\lambda^A:(z) = -31:\lambda^A\partial\lambda^A:(z)$$

where in the last step, sum over A is implicit and the sum over B gives 31. Now, using (11.5.6) with $\psi^2 = 2, k = 1, h(g) = 30$ (which is relevant for $SO(32)$), we get;

$$T^s_B(z) = -\frac{1}{(1+30)(2)}31:\lambda^A\partial\lambda^A:(z) = -\frac{1}{2}:\lambda^A\partial\lambda^A:(z)$$

So, we get (11.5.8) at the end.

For any Lie algebra g, there are bounds on central changes (**derive this**);

$$\text{rank}(g) \leq c^{g,h} \leq \dim(g)$$

11.5.2 Primary fields

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11.6 The bosonic construction and the toroidal compactification

If we have d non compact dimensions, then the l momenta are as follows;

$$(l_L^m, l_R^n)\ \ d\leq m\leq 25,\ \ d\leq n\leq 9$$

Using GSO projection on the right sector, (**find out how**) we get the same conditions on the lattice Γ;

$$\Gamma = \Gamma^*,\ \ l\circ l\in 2\mathbb{Z}$$

where $\circ$ is has signature of $(26-d, 10-d)$. For $d = 10$, all the signs in the $\circ$ inner product are positive, and thus, we have a Euclidean lattice. Even, self-dual Euclidean lattices exist only for dimensions that are multiples of 8 (**Find out its proof**). Since we want 16 dimensional lattices, we have the following even, self-dual lattices;

$$\Gamma_{16} = (n_1, ...n_{16}) \text{ or } \left(n_1 + \frac{1}{2}, ...n_{16} + \frac{1}{2}\right), \ \sum_j n_j \in 2\mathbb{Z} \tag{11.6.1}$$

$$\Gamma_8 \times \Gamma_8 \text{ where } \Gamma_8 = (n_1, ...n_8) \text{ or } \left(n_1 + \frac{1}{2}, ...n_8 + \frac{1}{2}\right), \ \sum_j n_j \in 2\mathbb{Z} \tag{11.6.2}$$

Now, we investigate the massless spectrum of the heterotic strings. Since we have 26 bosons in the left sector, just like the bosonic theory and we have at least two non-compact dimensions, the normal ordering constant in the left sector is -1. So, the massless states have $\partial X^\mu, \partial X^m$ and $e^{ik_L \cdot X_L}$ as the massless vertex operators which correspond to the following states;

$$\alpha^\mu_{-1}|k_L^2 = 0\rangle, \ \alpha^m_{-1}|k_L^2 = 0\rangle, \ |k_L^2 = 4/\alpha'\rangle \tag{11.6.3}$$

The first two of them have level 1 and the last one has level zero. Using (8.4.1), we see that if $k_R^2 = 0$ and $m^2 = 0$, then $N = 1$ and $k_L^2 = 0$ gives a massless state. Also, if $N = 0$ and $k_L^2 = 4/\alpha'$, then the state is again massless. If $k_L^2 = 4/\alpha'$ then $l_L^2 = \alpha'/2 \times 4/\alpha' = 2$. First state in (11.6.3) is in $\mathbf{8}_v$ representation of $SO(8)$. The second set of states has 16 different states and it lives in the Cartan subalgebra of gauge group because ∂X^m are 16 mutually commuting operators (**find more about its justification**). The OPE of these currents is as follows;

$$\partial X^m(z)\partial X^n(0) = \frac{1}{z^2}\delta^{mn} + ... \tag{11.6.4}$$

l_L^m are roots of the gauge group because they are charges of ∂X^m (**find its justification as well**).
The last state has $l_L^2 = 2$ where l_L is a 16 dimensional vector living in either Γ^{16} or $\Gamma^8 \times \Gamma^8$. From the $(n_1, ..., n_{16})$ Γ^{16} lattice, we see that the only way that we can have $l_L^2 = 2$ is by having $n_i^2 = n_j^2 = 1$ for some i, j where $i \neq j$. So, the possible integer Γ^{16} lattices are as follows;

$$(1, 1, \underbrace{0, ..., 0}_{14}) + \text{ permutations}, \ (1, -1, \underbrace{0, ..., 0}_{14}) + \text{ permutations}, \ (-1, -1, \underbrace{0, ..., 0}_{14}) + \text{ permutations}$$

These are nothing but the $SO(32)$ root lattices as shown in (11.4.20),(11.4.21) and (11.4.22) for $n = 16$. The half-integer Γ^{16} lattice doesn't show up in the massless spectrum of $SO(32)$ theory because in the $SO(32)$ theory, there are no massless states in the Ramond sector and this half-integer lattice corresponds to the Ramond sector (**find justification**).
However, this half-integer lattice is seen to have the following form;

$$(n_1, ..., n_{16}) + \underbrace{\left(\frac{1}{2}, ..., \frac{1}{2}\right)}_{16}, \ \sum_i n_i = 2\mathbb{Z}$$

So this lattice is just a sum of a spinor weight corresponding to 2^{15} spinor representation of $SO(32)$ and an arbitrary root of $SO(32)$. An arbitrary root means that it is an arbitrary integer-valued linear combination of the roots of $SO(32)$. Notice that we can replace an even number of $1/2$ in the weight above by $-1/2$ and replace n_i's in the corresponding positions by $n_i + 1$. Since an even number of 1's are added, the sum of all new n_i's is still even, and thus, the root vector with new n_i's is still an arbitrary root of $SO(32)$.
The integer lattices from $\Gamma^8 \times \Gamma^8$ that have $l_L^2 = 2$ are as follows;

$$(1, 1, \underbrace{0, ..., 0}_{6}) + \text{ permutations} \times (\underbrace{0, ..., 0}_{8}), \ (\underbrace{0, ..., 0}_{8}) \times (1, 1, \underbrace{0, ..., 0}_{6}) + \text{ permutations}$$

$$(1, -1, \underbrace{0, ..., 0}_{6}) + \text{ permutations} \times (\underbrace{0, ..., 0}_{8}), \ (\underbrace{0, ..., 0}_{8}) \times (1, -1, \underbrace{0, ..., 0}_{6}) + \text{ permutations}$$

$$(-1, -1, \underbrace{0, ..., 0}_{6}) + \text{ permutations} \times (\underbrace{0, ..., 0}_{8}), \ (\underbrace{0, ..., 0}_{8}) \times (-1, -1, \underbrace{0, ..., 0}_{6}) + \text{ permutations}$$

The following lattices;

$$(1,\underbrace{0,...,0}_{7})+\text{ permutations}\times(1,\underbrace{0,...,0}_{7})+\text{ permutations}$$

$$(1,\underbrace{0,...,0}_{7})+\text{ permutations}\times(-1,\underbrace{0,...,0}_{7})+\text{ permutations},\quad(-1,\underbrace{0,...,0}_{7})+\text{ permutations}\times(1,\underbrace{0,...,0}_{7})+\text{ permutations}$$

$$(-1,\underbrace{0,...,0}_{7})+\text{ permutations}\times(-1,\underbrace{0,...,0}_{7})+\text{ permutations}$$

are not allowed because the factor lattices are not allowed Γ^8 lattices as $\sum_{i=1}^{8} n_i$ should be even in each Γ^8 lattice (as mentioned in (11.6.2)). The allowed $\Gamma^8\times\Gamma^8$ are nothing but $SO(16)$ root lattices and thus the $\Gamma^8\times\Gamma^8$ lattice is $SO(16)\times SO(16)$ root lattice. However, that's not all. We didn't consider the half-integer $\Gamma^8\times\Gamma^8$ lattices yet. We can see that in the allowed lattices of this type, we can't have any n_i which isn't 0 or -1. The reason is that if some $n_i\geq 1$ then;

$$\left(n_i+\frac{1}{2}\right)^2\geq\frac{9}{4}$$

but this is already greater than 2 and thus, we can never have $l_L^2=2$ because the squares of the other components of the lattice will only contribute positively. A similar argument tells that n_i can't be less than -1. So, the allowed Γ^8 lattices are the ones where all n_i are zero or an even number of n_i's are -1 (to make the sum of all n_i's even). These lattices, combined with the integer lattices that we discussed before imply that the full 16-dimensional $\Gamma^8\times\Gamma^8$ lattice is an $E_8\times E_8$ lattice. Since both $SO(32)$ and $E_8\times E_8$ have longest roots of length 2 (**a bit ambiguity about $E_8\times E_8$**), we have $\hat{k}=k$ for the current algebras corresponding to these gauge groups and thus, comparing with (11.6.4), we see that $k=1$ for both of these current algebras. The following definitions are useful.

- The **root lattice** of a Lie algebra g is the set of all the integer-coefficient linear combinations of the roots of g. This lattice is denoted as Γ_g.
- Let r be a representation of g and λ be a weight vector of the r. Then, all vectors of the form $\lambda+v$ where $v\in\Gamma_g$ comprise what is known as a **weight sublattice** lattice. This lattice is dependent on the representation and thus, it is denoted as Γ_r. **Can we generate it by a single weight in a representation?**
- The union of Γ_r for all representations is called the **weight lattice** of g. It is denoted as Γ_w.

A useful result says that **(derive this)** for a simply laced algebra (recall that they don't include B_n algebras i.e. $SO(2n+1)$ and C_n algebras i.e. $USp(n)$) we have;

$$\Gamma_r\subset\Gamma_g^*\ \ \forall r$$

Moreover, we have;

$$\Gamma_w=\Gamma_g^*$$

This implies that for E_8 group, $\Gamma_g=\Gamma_g^*=\Gamma_w$ and thus, the weight and root lattices are the same for E_8. This is true for all simply laced Lie algebras whose root lattice is self-dual. **(Write about the left and right moving bosons).**

11.6.1 Toroidal compactification

Now, if we have $d<10$ non-compact dimensions, then the $l\circ l'$ inner product has $(26-d,10-d)$ signature. Just like the bosonic theory, we see that the inner product is invariant in $O(26-d,10-d,\mathbb{R})$ rotations but the mass formula and the constraint in (8.4.1) are invariant only under $O(26-d,\mathbb{R})\times O(10-d,\mathbb{R})$ rotations. Moreover, the compact momenta lattices are invariant under $O(26-d,10-d,\mathbb{Z})$ transformations. In other words, we can start from a reference lattice Γ_0 such that all compact directions are orthogonal and all radii are $SU(2)\times SU(2)$ radii. Then, the following equality is true among the lattices;

$$\Lambda'\Lambda\Lambda''\Gamma_0=\Lambda\Gamma_0$$

where we have;

$$\Lambda \in O(26-d, 10-d, \mathbb{R}), \;\; \Lambda' \in O(26-d, \mathbb{R}) \times O(10-d, \mathbb{R}), \;\; \Lambda'' \in O(26-d, 10-d, \mathbb{Z})$$

and therefore, the moduli space of the inequivalent theories is;

$$\frac{O(26-d, 10-d, \mathbb{R})}{O(26-d, \mathbb{R}) \times O(10-d, \mathbb{R}) \times O(26-d, 10-d, \mathbb{Z})}$$

where $O(26-d, 10-d, \mathbb{Z})$ elements act on elements of $O(26-d, 10-d, \mathbb{R})$ from the right. Now, we consider the massless spectrum of this theory. We can have the following three kinds of massless gauge bosons (**write about the non-gauge boson particles and the Ramond sector gauge boson**);

$$\alpha^m_{-1}\tilde{\psi}^\mu_{-1/2}|0\rangle, \quad m \in \{d, ..., 25\}$$

$$\alpha^\mu_{-1}\tilde{\psi}^m_{-1/2}|0\rangle, \quad m \in \{d, ..., 9\}$$

$$e^{ik_L\cdot X_L}\tilde{\psi}^\mu|0\rangle \quad m \in \{d, ..., 9\}, \;\; l_L^2 = 2$$

(Write about the origin of these gauge bosons i.e. commuting currents, Kaluza Klein, and anti-symmetric tensor). The vertex operators of first type of gauge bosons are $\partial X^m\tilde{\psi}^\mu$, the operators for second type of bosons are $\psi X^\mu\tilde{\psi}^m$ and the operator for the last one are just $e^{ik_L\cdot X_L}\tilde{\psi}^\mu$. There are $26-d$ bosons of the first type, $10-d$ bosons of the second type, and $10-d$ bosons of the last type. Since all of these states have level $\tilde{N} = 1$, we can't have $l_R^2 \neq 0$ for a massless state because the mass of such a state would be at least $l_R^2/2$ (**derive this**). For generic points, $l_L^2 \neq 2$ and thus, the gauge group is $U(1)^{26-d+10-d} = U(1)^{36-2d}$.

11.6.2 Supersymmetry and BPS states

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12 Chapter 12: Superstring interactions

12.1 Low energy supergravity

12.1.1 Type IIA superstring

We will show the connection between 11-dimensional SUGRA and type II A superstrings. The spectrum (**find justification**) of 11-dimensional SUGRA (graviton multiplet) is an $SO(9)$ traceless symmetric matrix (graviton), a three form A_3 and a gravito-vector representation. Both the fermions and bosons have 128 states as follows;

$$\text{Bosons: } \underbrace{\frac{9\times 8}{2} - 1}_{\text{graviton}} + \underbrace{\binom{9}{3}}_{A_3} = 44 + 84 = 128$$

$$\text{Fermions: } \underbrace{9\times 16}_{\text{indices}} - \underbrace{16}_{\text{const.}} = 144 - 16 = 128$$

Find out more about the trace constraint. For the low-energy bosonic action, we have the following for 11 D SUGRA;

$$S_{11} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa^2}\int A_3\wedge F_4\wedge F_4 \tag{12.1.1}$$

Now, for dimensional reduction, we choose the standard reduced metric form as follows;

$$G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dx^{10} + A_\mu dx^\mu),\ \ \mu = 0,1,...9$$

$$= (G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu)dx^\mu dx^\nu + 2e^{2\sigma}A_\mu dx^\mu dx^{10} + e^{2\sigma}dx^{10}dx^{10}$$

where $G_{10,10} = e^{2\sigma}$. We now calculate the action that results from (12.1.1) under the dimensional reduction quoted above. The 11-dimensional R is calculated to be (**fill in details**);

$$R^{11} = R^{10} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}|F_2|^2$$

The block form of the metric is given as follows;

$$\begin{pmatrix}\tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu\\ e^{2\sigma}A_\mu & e^{2\sigma}\end{pmatrix}\ \ \text{where } \tilde{G}_{\mu\nu} = G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu$$

The determinant of a block matrix is given as follows;

$$\det\begin{bmatrix}A & B\\ C & D\end{bmatrix} = (\det A)\det(D - CA^{-1}B) \Rightarrow \det\begin{bmatrix}\tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu\\ e^{2\sigma}A_\mu & e^{2\sigma}\end{bmatrix} = e^{2\sigma}\tilde{G} - e^{4\sigma}\tilde{G}\tilde{G}^{\mu\nu}A_\mu A_\nu$$

We can calculate $\tilde{G}^{\mu\nu}$ (i.e. the inverse of $\tilde{G}_{\mu\nu}$) by assuming a form;

$$\tilde{G}^{\mu\nu} = \alpha G^{\mu\nu} + \beta A^\mu A^\nu$$

and we readily get;

$$\alpha = 1,\ \ \beta = -\frac{e^{2\sigma}}{1 + e^{2\sigma}(A)^2},\ \ (A)^2 = A_\mu A^\mu \Rightarrow \tilde{G}^{\mu\nu} = G^{\mu\nu} - \frac{e^{2\sigma}}{1 + e^{2\sigma}(A)^2}A^\mu A^\nu$$

Using this expression, the required determinant becomes;

$$\det\begin{bmatrix}\tilde{G}_{\mu\nu} & e^{2\sigma}A_\mu\\ e^{2\sigma}A_\mu & e^{2\sigma}\end{bmatrix} = e^{2\sigma}\tilde{G}\left[1 - e^{2\sigma}\left(G^{\mu\nu} - \frac{e^{2\sigma}}{1 + e^{2\sigma}(A)^2}A^\mu A^\nu\right)A_\mu A_\nu\right] = \frac{e^{2\sigma}\tilde{G}}{1 + e^{2\sigma}(A)^2}$$

We now evaluate $\tilde{G}$. $\tilde{G}_{\mu\nu}$ is given as follows;

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu = G_{\mu\alpha}\left(\delta^\alpha_\nu + e^{2\sigma}A^\alpha A_\nu\right) \Rightarrow \tilde{G} = G\det\left(\mathbb{I} + M\right)\quad \text{where } M^\alpha_\beta = e^{2\sigma}A^\alpha A_\beta$$

Now, we use the following identity;

$$\det(A) = e^{\mathrm{tr}(\ln A)} \Rightarrow \det(\mathbb{I} + M) = e^{\mathrm{tr}(\ln(\mathbb{I}+M))}$$

Now, we have;

$$(\ln(\mathbb{I} + M))^{\alpha}_{\beta} = \sum_{j=1}^{\infty} (-1)^j \frac{(M^j)^{\alpha}_{\beta}}{j} \Rightarrow \mathrm{tr}\,(\ln(\mathbb{I} + M)) = \sum_{j=1}^{\infty} (-1)^j \frac{\mathrm{tr}\, M^j}{j}$$

Now, we have;

$$(M^j)^{\alpha}_{\beta} = e^{2j\sigma} A^{\alpha} A_{\mu_1} A^{\mu_1} A_{\mu_2} ... A^{\mu_{j-2}} A_{\mu_{j-1}} A^{\mu_{j-1}} A_{\beta} = A^{2(j-1)} A^{\alpha} A_{\beta} \Rightarrow \mathrm{tr}(M^j) = e^{2j\sigma} A^{2j}$$

which gives us the following;

$$\mathrm{tr}\,(\ln(\mathbb{I} + M)) = \sum_{j=1}^{\infty} (-1)^j \frac{e^{2j\sigma} A^{2j}}{j} = \ln(1 + e^{2\sigma} A^2) \Rightarrow \det(\mathbb{I} + M) = 1 + e^{2\sigma} A^2 \Rightarrow \tilde{G} = G(1 + e^{2\sigma} A^2)$$

Therefore, we get (I loved this calculation);

$$\det \begin{bmatrix} \tilde{G}_{\mu\nu} & e^{2\sigma} A_{\mu} \\ e^{2\sigma} A_{\mu} & e^{2\sigma} \end{bmatrix} = \frac{e^{2\sigma} G(1 + e^{2\sigma}(A)^2)}{1 + e^{2\sigma}(A)^2} = e^{2\sigma} G$$

Since all the fields are independent of x^{10} direction, the x^{10} integral just gives $2\pi R$ and thus, the Ricci term in (12.1.1) becomes;

$$\frac{1}{2\kappa^2} \int d^{11}x \sqrt{-G} R = \frac{2\pi R}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{\sigma} \left(R - 2e^{-\sigma} \nabla^2 e^{\sigma} - \frac{1}{4} e^{2\sigma} |F_2|^2 \right)$$

$$= \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-G} \left(e^{\sigma} R - 2\nabla^2 e^{\sigma} - \frac{1}{4} e^{3\sigma} |F_2|^2 \right)$$

(**where would this sigma term go?**) where $\kappa_0^2 = \kappa^2 / 2\pi R$. Now, we do the $|F_4|^2$ term in (12.1.1). The term is hard to manipulate. It goes as follows;

$$-\frac{1}{4.4!\kappa_0^2} \int d^{10}x \sqrt{-G}\, e^{\sigma} G^{M_1 N_1} ... G^{M_4 N_4} F_{M_1 ... M_4} F_{N_1 ... N_4}$$

$$= -\frac{1}{96\kappa_0^2} \int d^{10}x \sqrt{-G}\, e^{\sigma} G^{M_1 N_1} ... G^{M_4 N_4} F_{M_1 ... M_4} F_{N_1 ... N_4}$$

Now, the dot product is evaluated as follows;

$$G^{\mu_1\nu_1} ... G^{\mu_4\nu_4} F_{\mu_1 ... \mu_4} F_{\nu_1 ... \nu_4} + 8 G^{\mu_1 d} ... G^{\mu_4\nu_4} F_{\mu_1 ... \mu_4} F_{d ... \nu_4}$$

$$+ 4 G^{dd} ... G^{\mu_4\nu_4} F_{d ... \mu_4} F_{d ... \nu_4} + 6 G^{d\nu_1} G^{\mu_2 d} G^{\mu_2\nu_3} G^{\mu_4\nu_4} F_{d\mu_2\mu_3\mu_4} F_{\nu_1 d \nu_3 \nu_4} \tag{12.1.2}$$

Consider the second term above. One kind of general term for this is;

$$G^{\mu_1\nu_1} ... G^{\mu_j d} ... G^{\mu_4\nu_4} F_{\mu_1 ... \mu_j ... \mu_4} F_{\nu_1 ... d ... \nu_4}$$

Interchange the indices μ_1 and μ_j, also rename the index ν_1 to ν_j. Now, we can do some permutations to interchange the first and jth index on both F's, and thus, we get a factor of unity by doing this. So, we can transform such general terms for any j to the form written above. There are four terms like this (as $j = 1, 2, 3, 4$). Another kind of general term is as follows;

$$G^{\mu_1\nu_1} ... G^{d\nu_j} ... G^{\mu_4\nu_4} F_{\mu_1 ... d ... \mu_4} F_{\nu_1 ... \nu_j ... \nu_4}$$

In this term, we do interchange $\mu_k \leftrightarrow \nu_k$ for $k \neq j$ and set $\nu_j = \mu_j$. Moreover, by using the symmetry of G, we can get this kind of term to be like the general term of the first kind. Again, we have four terms like this. So, we have eight terms in total. Now, we can consider the case where one of μ_j's is d and one of ν_k's is d. One case arises when $j = k$. There are four terms like this. Again, we can show that all of these terms are equal. That gives us the third term in (12.1.2). The last case arises when $j \neq k$. The general term, in this case, is as follows;

$$G^{\mu_1\nu_1}...G^{d\nu_j}...G^{\mu_k d}...G^{\mu_4\nu_4}F_{\mu_1...d...\mu_k...\mu_4}F_{\nu_1...\nu_j...d...\nu_4}$$

We interchange d and μ_1 indices in the first F and similarly, we will interchange d and ν_2 in the second F. This may give us a power of -1. This power is $2j-3+2(k-1)-3$ which is even and thus, no minus factors appear. Then, rename two indices as $\mu_1 \to \mu_j$, $\nu_2 \to \nu_k$ and interchange the indices $\nu_j \leftrightarrow \nu_1$, $\nu_k \leftrightarrow \nu_2$. After that, we interchange the indices in the F's again. This time, interchange d with μ_2 and ν_j with ν_1. This may again give us a power of -1. This power is $2(k-1)-3+2j-3$ which is again even. So, after this procedure, this general term becomes;

$$G^{d\nu_1}G^{\mu_2 d}G^{\mu_3\nu_3}G^{\mu_4\nu_4}F_{d\mu_2\mu_3\mu_4}F_{\nu_1 d\nu_3\nu_4}$$

Moreover, there are $\binom{4}{2} = 6$ terms like this and thus we have a factor of 6 in the last term of (12.1.2).
The first term in (12.1.2) gives us a term proportional to $|F_4|^2$ and thus, we get the term;

$$-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\, e^{\sigma}\frac{1}{4!}G^{\mu_1\nu_1}...G^{\mu_4\nu_4}F_{\mu_1...\mu_4}F_{\nu_1...\nu_4} = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\, e^{\sigma}|F_4|^2$$

The third term in (12.1.2) gives us the following term;

$$-\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\, e^{\sigma}\frac{4}{4!}(e^{-2\sigma}+A^2)G^{\mu_2\nu_2}G^{\mu_3\nu_3}G^{\mu_4\nu_4}F_{d\mu_2\mu_3\mu_4}F_{d\nu_2\nu_3\nu_4}$$

$$= -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\, e^{-\sigma}|F_3|^2 - \frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\, e^{\sigma}A^2|F_3|^2$$

(**what about this extra term?**). The expression for G^{dd} is found from the expression given below. Now, we consider the second term in (12.1.2). For that, we will need the value of $G^{d\mu}$. To do this, we will need the formula for the inverse of a matrix in block form. We won't need the general formula but just a special case as follows;

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+\frac{1}{k}A^{-1}bb^TA^{-1} & -\frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix} \quad \text{where } k = c - b^TA^{-1}b$$

We won't need the full matrix but just $G^{d\mu}$. We do it as follows;

$$G^{d\mu} = \left(e^{2\sigma}\left[G^{\mu\nu} - \frac{e^{2\sigma}}{1+e^{2\sigma}A^2}A^\mu A^\nu\right]A_\mu A_\nu - 1\right)^{-1}\left[\left(G^{\mu\nu} - \frac{e^{2\sigma}}{1+e^{2\sigma}A^2}A^\mu A^\nu\right)A_\nu\right] = -A^\mu$$

We can also calculate $1/k$ as follows;

$$\frac{1}{k} = e^{-2\sigma} + A^2$$

Now, the second term in (12.1.2) is as follows;

$$-8G^{\mu_1\nu_1}...G^{\mu_4\nu_4}(F_4)_{\mu_1...\mu_4}A_{\nu_1}(F_3)_{\nu_2\nu_3\nu_4} = -2G^{\mu_1\nu_1}...G^{\mu_4\nu_4}(F_4)_{\mu_1...\mu_4}(A\wedge F_3)_{\nu_1\nu_2\nu_2\nu_3}$$

where in the last line, we used the fact that only the antisymmetric part of AF_3 factor will survive this summation. The last term in (12.1.2) is as follows;

$$6G^{\mu_1\nu_1}G^{\mu_2\nu_2}G^{\mu_3\nu_3}G^{\mu_4\nu_4}A_{\mu_1}(F_3)_{\mu_2\mu_3\mu_4}A_{\nu_1}(F_3)_{\nu_2\nu_3\nu_4}$$

(**This contains the A^2 term and find out the anti-symmetric argument**). The Chern Simons terms are done as follows;

$$-\frac{1}{12\kappa^2}\frac{11!}{3!(4!)^2}\int dx^{\mu_1}...dx^{\mu_{11}}A_{[[\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6\mu_7]}F_{\mu_8\mu_9\mu_{10}\mu_{11}]}$$

Now, we concentrate on the index which is d. This can be in A (3 cases) or in F (8 cases). The dx^d integration gives $2\pi R$. Thus, we get the following;

$$-\frac{2\pi R}{12\kappa^2}\frac{11!}{3!(4!)^2}\left[3\frac{2!(4!)^2}{10!}\int A_2\wedge F_4\wedge F_4 + 8.\frac{4!(3!)^2}{10!}\int A_3\wedge F_4\wedge F_3\right]$$

$$= -\frac{11}{12\kappa_0^2}\left[\int A_2 \wedge F_4 \wedge F_4 + 2\int A_3 \wedge F_4 \wedge F_3\right]$$

We prove that all of these three terms are equal which is shown as follows;

$$0 = \int d(A_2 \wedge F_4 \wedge A_3) = \int (A_2 \wedge F_4 \wedge dA_3 + dA_2 \wedge F_4 \wedge A_3) = \int (A_2 \wedge F_4 \wedge F_4 + F_3 \wedge F_4 \wedge A_3)$$

$$\Rightarrow \int A_2 \wedge F_4 \wedge F_4 = -\int F_3 \wedge F_4 \wedge A_3 = \int A_3 \wedge F_4 \wedge F_3$$

where I used the following identities;

$$d^2 = 0 \Rightarrow d(F_p) = d^2 A_{p-1} = 0$$

$$\omega_p \wedge \omega_q = (-1)^{pq}\omega_q \wedge \omega_p$$

where ω_p is any p form. So, the Chern Simons terms become term becomes;

$$= -\frac{11}{4\kappa_0^2}\int A_2 \wedge F_4 \wedge F_4 = -\frac{11}{4\kappa_0^2}\int A_3 \wedge F_4 \wedge F_3$$

(Why is this 11 term appearing?). So, the action in ten dimensions is as follows;

$$S = S_1 + S_2 + S_3$$

where

$$S_1 = \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}\left(e^{\sigma}R - \frac{1}{4}e^{3\sigma}|F_2|^2\right)$$

$$S_2 = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\left(e^{-\sigma}|F_3|^2 + e^{\sigma}|\tilde{F}_4|^2\right)$$

$$S_3 = -\frac{1}{4\kappa_0^2}\int A_2 \wedge F_4 \wedge F_4 = -\frac{1}{4\kappa_0^2}\int A_3 \wedge F_3 \wedge F_4$$

where $\tilde{F}_4 = dF_3 - A_1 \wedge F_3$. We now show that the Chern-Simons term is gauge invariant;

$$A_3 \wedge F_3 \wedge F_4 \to (A_3 + d\lambda_2) \wedge F_3 \wedge F_4 = A_3 \wedge F_3 \wedge F_4 + d(\lambda_2 \wedge F_3 \wedge F_4) = A_3 \wedge F_3 \wedge F_4 + \text{total derivative}$$

Now, we talk about gauge invariance term dependent on $\tilde{F}$. We see that it is gauge invariant under the following transformation;

$$\delta\tilde{F}_4 = d(d\lambda_2) = 0$$

However, in the transformation of A_1, we have;

$$\delta\tilde{F}_4 = -d\lambda_0 \wedge F_3 = -d(\lambda_0 \wedge F_3)$$

which isn't zero generally. To treat this, we need A_3 to be transformed in this gauge transformation as well. The required transformation is as follows;

$$\delta A_3 = \lambda_0 \wedge F_3$$

This additional transformation comes from the reparametrization of x^{10} (**motivated but justify more**). We see that $\tilde{F}_4$ satisfies a non-trivial Bianchi identity as follows;

$$d\tilde{F}_4 = -d(A_1 \wedge F_3) = -dA_1 \wedge F_3 = -F_2 \wedge F_3$$

We want to find the relation between σ and the dilaton ϕ now. We redefine the metric as follows;

$$G^{\text{old}}_{\mu\nu} = e^{-\sigma}G^{\text{new}}_{\mu\nu}$$

The old and new superscripts will be removed now. This changes G and R as follows;

$$G \to e^{-10\sigma}G \Rightarrow \sqrt{-G} \to e^{-5\sigma}\sqrt{-G}$$

$$R \to e^{\sigma}R - \frac{9}{2}e^{3\sigma}\nabla_\mu\nabla^\mu e^{-2\sigma}$$

Now, the R becomes as follows in this transformation;

$$\sqrt{-G}e^{\sigma}R \to \sqrt{-G}e^{-3\sigma}R$$

However, we want this to be as follows;

$$\sqrt{-G}e^{-2\phi}R \Rightarrow \sigma = \frac{2\phi}{3}$$

Now, can expand the R transformation as follows;

$$e^{\sigma}R - \frac{9}{2}e^{\sigma}\left[4\partial_\mu\sigma\partial^\mu\sigma - 2\nabla_\mu\partial^\mu\sigma\right] = e^{\sigma}R - \frac{9}{2}\left(6e^{\sigma}\partial_\mu\sigma\partial^\mu\sigma - 2\nabla_\mu(e^{\sigma}\partial^\mu\sigma)\right) = e^{\sigma}R - 27\partial_\mu\sigma\partial^\mu\sigma + \text{derivative}$$
$$= e^{-2\phi/3}R - 12e^{\sigma}\partial_\mu\phi\partial^\mu\phi + \text{derivative}$$

(Extra factor of 3 and a minus sign!!!). In addition, we have the following;

$$|F_p|^2 = G^{\mu_1\nu_1}...G^{\mu_p\nu_p}F_{\mu_1...\mu_p}F_{\nu_1...\nu_p} \to e^{p\sigma}G^{\mu_1\nu_1}...G^{\mu_p\nu_p}F_{\mu_1...\mu_p}F_{\nu_1...\nu_p} = e^{p\sigma}|F_p|^2$$

So, the actions in new metric become as follows;

$$S_1 = \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\phi}|F_2|^2\right)$$
$$S_2 = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}(|F_3|^2 + e^{2\phi}|\tilde{F}_4|^2)$$

The Chern Simons term doesn't change because it is independent of metric. Now, in type II A theory, the bosonic spectrum is as follows;

$$(\text{NS+}, \text{NS+}):\ [0] + [2] + (2)$$
$$(\text{R+}, \text{R}-):\ [1] + [3]$$

So, in the (NS+, NS+) sector, we have a dilaton, a graviton and a two form field (which we will now call B_2 and its field strength is called H_3). In (R+, R−) sector, we have a one-form field and a three-form field which we will call C_1, C_3 and their corresponding field strengths are F_2, F_4. So, we can arrange the three actions above as NS sector action and R sector action as follows;

$$S_{\text{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\phi\right) \tag{12.1.3}$$

$$S_{\text{R}} = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}(|F_2|^2 + |\tilde{F}_4|^2) \tag{12.1.4}$$

$$S_{\text{CS}} = -\frac{1}{4\kappa_0^2}\int\ B_2\wedge F_4\wedge F_4 \tag{12.1.5}$$

(Write the rest of massless II A section).

12.1.2 Massive type IIA supergravity

The bosonic field content of the massless IIA theory is as follows;

$$\underbrace{G_{\mu\nu},\ H_3 = dB_2,\ \phi}_{\text{(NS+,NS+) sector}}\quad \underbrace{F_2 = dC_1,\ F_4 = dC_3}_{\text{(R+,R−) sector}}\quad \underbrace{F_6 = dC_5,\ F_8 = dC_7}_{\text{Poincare duals}}$$

Following the pattern, we may guess that there might be a F_{10} field strength (**talk more about it**). If it is there, then the EOM for this field strength is;

$$d * F_{10} = 0 \Rightarrow *F_{10} = \text{constant}$$

where the last step follows because $*F_{10}$ is a scalar. The action that involves this new field is as follows (**try to justify it**);

$$S_{IIA}^{\text{massive}} = \tilde{S}_{IIA} - \frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}M^2 + \frac{1}{4\kappa_{10}^2}\int MF_{10} \tag{12.1.6}$$

where $\tilde{S}_{IIA}$ is the sum of (12.1.3), (12.1.4) and (12.1.5) but with the following changes;

$$F_2 \to F_2 + MB_2,\ \ F_4 \to F_4 + \frac{1}{2}MB_2\wedge B_2 \Rightarrow \tilde{F}_4 \to \tilde{F}_4 + \frac{1}{2}MB_2\wedge B_2$$

We see that M is just an auxiliary field as it has no derivatives in (12.1.6).

12.1.3 Type IIB supergravity

The field content of type IIB is as follows;

$$\underbrace{G_{\mu\nu},\ H_3(=dB_2),\ \ \phi}_{\text{(NS+,NS+) sector}}\quad \underbrace{F_1 = dC_0\ \ ,F_3 = dC_2,\ \ F_5 = dC_4}_{\text{(R}\pm\text{,R}\pm\text{) sectors}}\quad \underbrace{F_5 = dC_4\ \ ,F_7 = dC_6,\ \ F_9 = dC_8}_{\text{Poincare duals}}$$

Since we are working in ten dimensions which is 2 mod 4, we can impose one of the following conditions;

$$*F_5 = \pm F_5$$

as we saw in Appendix B. Moreover, since $|F_5|^2 = 0$, we can't have this in the action. The following actions can give us a low energy theory for IIB (**justify this**);

$$S_{\text{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\phi\right) \tag{12.1.7}$$

$$S_{\text{R}} = -\frac{1}{4\kappa_0^2}\int d^{10}x\sqrt{-G}(|F_1|^2 + |\tilde{F}_3|^2 + |\tilde{F}_5|^2) \tag{12.1.8}$$

$$S_{\text{CS}} = -\frac{1}{4\kappa_0^2}\int\ \ C_4\wedge H_3\wedge F_3 \tag{12.1.9}$$

where we have;

$$\tilde{F}_3 = F_3 - C_0\wedge H_3,\ \ \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$$

From these definitions, we can deduce the EOM and Bianchi identity for $\tilde{F}_5$ as follows;

$$d*\tilde{F}_5 = d*F_5 - \frac{1}{2}d*(C_2\wedge H_3) + \frac{1}{2}d*(B_2\wedge F_3) = -\frac{1}{2}d*(C_2\wedge H_3) + \frac{1}{2}d*(B_2\wedge F_3) = H_3\wedge F_3$$

justify the last step. Therefore, $d\tilde{F}_5 = d*\tilde{F}_5$ and thus, $\tilde{F}_5 = *\tilde{F}_5$ is consistent with the EOM and Bianchi identity but it isn't implied by them (as $\tilde{F}_5$ and $*\tilde{F}_5$ might differ by an exact form). If we impose this condition on the action, the EOM of $\tilde{F}_5$ won't change but the EOMs for other fields will change. So, we impose this self-duality condition on the solutions.
We now investigate the symmetry of the low-energy action. Define the following;

$$G_{E\mu\nu} = e^{-\phi/2}G_{\mu\nu},\quad \tau = C_0 + ie^{-\phi},\quad \mathcal{M}_{ij} = \frac{1}{\text{Im}(\tau)}\begin{bmatrix}|\tau|^2 & -\text{Re}(\tau)\\ -\text{Re}(\tau) & 1\end{bmatrix},\ \ F_3^i = \begin{pmatrix}H_3\\ F_3\end{pmatrix}$$

This implies that;

$$\text{Re}(\tau) = C_0,\ \ \text{Im}(\tau) = e^{-\phi}$$

We now derive a couple of results. The first on is as follows;

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3 = -\frac{1}{4\kappa_{10}^2}\int (C_4\wedge H_3\wedge F_3 - C_4\wedge F_3\wedge H_3) = -\frac{1}{4\kappa_{10}^2}\int\left(C_4\wedge F_3^1\wedge F_3^2 - C_4\wedge F_3^2\wedge F_3^1\right)$$

$$= -\frac{\epsilon_{ij}}{4\kappa_{10}^2}\int\left(C_4\wedge F_3^i\wedge F_3^j\right) \tag{12.1.10}$$

The second one is as follows;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(-\frac{1}{4}|\tilde{F}_5|^2\right) = -\frac{1}{8\kappa_{10}^2}\int d^{10}x\ e^{-5\phi/2}\sqrt{-G}\left(\tilde{F}_5\right)_{\alpha_1\ldots\alpha_5}\left(\tilde{F}_5\right)_{\beta_1\ldots\beta_5}G_E^{\alpha_1\beta_1}\ldots G_E^{\alpha_5\beta_5}$$

$$= -\frac{1}{8\kappa_{10}^2}\int d^{10}x\ e^{-5\phi/2}\sqrt{-G}\left(\tilde{F}_5\right)_{\alpha_1\ldots\alpha_5}\left(\tilde{F}_5\right)_{\beta_1\ldots\beta_5}e^{5\phi/2}\ G^{\alpha_1\beta_1}\ldots G^{\alpha_5\beta_5}$$

$$= -\frac{1}{8\kappa_{10}^2}\int d^{10}x\ \sqrt{-G}\left(\tilde{F}_5\right)_{\alpha_1\ldots\alpha_5}\left(\tilde{F}^5\right)_{\beta_1\ldots\beta_5} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\ \sqrt{-G}\left(\frac{1}{2}|\tilde{F}_5|^2\right) \tag{12.1.11}$$

where the last $|\tilde{F}_5|^2$ uses $G_{\mu\nu}$ instead of $G_{E\mu\nu}$ to contract the indices. Now, we derive another result.

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(-\frac{\mathcal{M}_{ij}}{2}F_3^i.F_3^j\right) = -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-5\phi/2}\sqrt{-G}\mathcal{M}_{ij}\left(F_3^i\right)_{\alpha_1\ldots\alpha_3}\left(F_3^j\right)_{\beta_1\ldots\beta_3}e^{3\phi/2}G^{\alpha_1\beta_1}\ldots G^{\alpha_3\beta_3}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}\left(F_3^i\right)_{\alpha_1\ldots\alpha_3}\left(F_3^j\right)_{\beta_1\ldots\beta_3}G^{\alpha_1\beta_1}\ldots G^{\alpha_3\beta_3} = -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}F_3^i.F_3^j$$

where in the last step, G metric is used instead of G_E to contract the indices. We manipulate this final expression as follows;

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}F_3^i.F_3^j = -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\left(\frac{|\tau|^2}{\mathrm{Im}(\tau)}|H_3|^2 + \frac{1}{\mathrm{Im}(\tau)}|F_3|^2 - \frac{2\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}H_3.F_3\right)$$

$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}xe^{-\phi}\sqrt{-G}\mathcal{M}_{ij}F_3^i.F_3^j = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left((C_0^2 + e^{-2\phi})|H_3|^2 + |F_3|^2 - 2C_0H_3.F_3\right)$$

$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right) + -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(C_0^2|H_3|^2 + |F_3|^2 - 2C_0H_3.F_3\right)$$

$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right) + -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}|F_3 - C_0\wedge H_3|^2$$

$$= \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right) + -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}|\tilde{F}_3|^2$$

Therefore, we have;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(-\frac{\mathcal{M}_{ij}}{2}F_3^i.F_3^j\right) = \frac{1}{2\kappa_{10}^2}\int d^{10}xe^{-2\phi}\sqrt{-G}\left(-\frac{1}{2}|H_3|^2\right) + -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}|\tilde{F}_3|^2 \quad (12.1.12)$$

Another result is derived as follows;

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{(\mathrm{Im}\tau)^2} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\, e^{-5\phi/2}\sqrt{-G}\frac{e^{\phi/2}G^{\mu\nu}\left(\partial_\mu C_0 + i\partial_\mu\phi e^{-\phi}\right)\left(\partial_\mu C_0 - i\partial_\mu\phi e^{-\phi}\right)}{e^{-2\phi}}$$

$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\, e^{-5\phi/2}\sqrt{-G}\frac{e^{\phi/2}|F_1|^2 + e^{-3\phi/2}\partial_\mu\phi\partial^\mu\phi}{e^{-2\phi}}$$

$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}|F_1|^2 + \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}e^{-2\phi}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) \quad (12.1.13)$$

Before deriving the last result, we use the fact that in $n \neq 2$ dimensions, if two metrics are related as

$$\tilde{g} = e^{2\phi}g$$

then the Ricci tensors are related as follows;

$$\tilde{R} = e^{-2\phi}\left(R - \frac{4(n-1)}{n-2}e^{-(n-2)\phi/2}g^{\mu\nu}\nabla_\mu\nabla_\nu e^{(n-2)\phi/2}\right)$$

also, recall that for a scalar ϕ, $\nabla_\mu\phi = \partial_\mu\phi$. Plugging in $n = 10$ and rescaling ϕ as $\phi \to -\phi/4$ (to match the relation between G and G_E that we have);

$$R_E = e^{\phi/2}\left(R - \frac{9}{2}e^{\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi}\right) = e^{\phi/2}R - \frac{9}{2}e^{3\phi/2}G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi}$$

Now, we do the following manipulation;

$$G^{\mu\nu}\nabla_\mu\nabla_\nu e^{-\phi} = G^{\mu\nu}\nabla_\mu\left(-\partial_\nu\phi e^{-\phi}\right) = -G^{\mu\nu}\left(\partial_\mu\partial_\nu\phi - \Gamma^\alpha_{\mu\nu}\partial_\alpha\phi\right)e^{-\phi} + G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi e^{-\phi}$$

$$= -G^{\mu\nu}\left(\nabla_\mu\nabla_\nu\phi\right)e^{-\phi} + \partial_\mu\phi\partial^\mu\phi e^{-\phi}$$

Thus, the expression of R_E becomes;

$$R_E = e^{\phi/2}\left[R + \frac{9}{2}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]$$

Now, we derive the last result that we need;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G_E}R_E = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e^{-5\phi/2}\sqrt{-G}e^{\phi/2}\left[R + \frac{9}{2}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right]$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}\left[R - \frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right] + \frac{9}{4\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$$

The integrand of the last term can be written as follows;

$$G^{\mu\nu}\nabla_\mu\left(e^{-2\phi}\nabla_\nu\phi\right) = G^{\mu\nu}\left(-2\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right) + e^{-2\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$$
$$\Rightarrow e^{-2\phi}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi = G^{\mu\nu}\nabla_\mu\left(e^{-2\phi}\nabla_\nu\phi\right) + 2G^{\mu\nu}\left(\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right) = \nabla^\mu\left(e^{-2\phi}\nabla_\nu\phi\right) + 2G^{\mu\nu}\left(\partial_\mu\phi e^{-2\phi}\partial_\nu\phi\right)$$

The first term in this expression will give a surface term and thus, we omit this term. Therefore, we get;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G_E}R_E = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}\left[R - \frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right] + \frac{9}{2\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$= \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}\left[R + \frac{9}{2}\partial_\mu\phi\partial^\mu\phi\right] \tag{12.1.14}$$

Note that the dilaton kinetic term from the sum of (12.1.13) and (12.1.14) is;

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\,e^{-2\phi}\sqrt{-G}\left(4\partial_\mu\phi\partial^\mu\phi\right)$$

which is exactly the term in (12.1.7). Now, using (12.1.10) to (12.1.14), it is easily seen that the following action gives the same action as the sum of (12.1.7), (12.1.8) and (12.1.9);

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(R_E - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\text{Im }\tau)^2} - \frac{\mathcal{M}_{ij}}{2}F_3^i.F_3^j - \frac{1}{4}|\tilde{F}_5|^2\right) - \frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4\wedge F_3^i\wedge F_3^j \tag{12.1.15}$$

We will now show that this action is invariant under the following $SL(2,\mathbb{R})$ transformations;

$$\tau\to\tau' = \frac{a\tau+b}{c\tau+d}\;\;(ad-bc=1),\;\; F_3^i\to F_3^{i'} = \Lambda^i_jF_3^j\;\;\Lambda = \begin{bmatrix} d & c\\ b & a\end{bmatrix}\in SL(2,\mathbb{R}) \tag{12.1.16}$$

with the other fields not changing. We show the invariance by showing the invariance of individual terms. We start with the following;

$$\partial_\mu\bar{\tau}'\partial^\mu\tau' = \partial_\mu\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)\partial^\mu\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)\partial_\mu\bar{\tau}(ad-bc)\partial^\mu\tau}{|c\tau+d|^4} = \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{|c\tau+d|^4}$$

Moreover, we have;

$$\text{Im}(\tau') = \frac{1}{2i}\left(\tau'-\bar{\tau}'\right) = \frac{1}{2i}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{1}{2i}\frac{(ad-bc)\tau-(ad-bc)\bar{\tau}}{|c\tau+d|^2} = \frac{1}{2i}\frac{\tau-\bar{\tau}}{|c\tau+d|^2} = \frac{\text{Im}(\tau)}{|c\tau+d|^2}$$
$$\Rightarrow (\text{Im}(\tau'))^2 = \frac{(\text{Im}(\tau))^2}{|c\tau+d|^4}$$

Therefore, we have;

$$\frac{\partial_\mu\bar{\tau}'\partial^\mu\tau'}{(\text{Im}(\tau'))^2} = \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{(\text{Im}(\tau))^2}$$

We now see how $\mathcal{M}_{ij}$ transforms. For this purpose, we first calculate the following;

$$\Lambda = \begin{bmatrix} d & c\\ b & a\end{bmatrix}\Rightarrow\Lambda^{-1} = \begin{bmatrix} a & -c\\ -b & d\end{bmatrix}\Rightarrow(\Lambda^{-1})^T = \begin{bmatrix} a & -b\\ -c & d\end{bmatrix}$$

$$\Rightarrow (\Lambda^T)^{-1}\mathcal{M}\Lambda^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} a & -c \\ -b & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 M_{11} - 2abM_{12} + b^2 M_{22} & -acM_{11} + (ad+bc)M_{12} - bdM_{22} \\ -acM_{11} + (ad+bc)M_{12} - bdM_{22} & c^2 M_{11} - 2cdM_{12} + d^2 M_{22} \end{bmatrix}$$

where we used the fact that $M_{12} = M_{21}$. We see that $(\Lambda^{-1})^T\mathcal{M}\Lambda^{-1}$ is symmetric too. We now claim that $\mathcal{M}$ transforms as follows;

$$\mathcal{M}' = (\Lambda^{-1})^T\mathcal{M}\Lambda^{-1}$$

We now prove this claim. For this purpose, we see the following;

$$\text{Re }(\tau') = \frac{1}{2}\left(\frac{a\tau+b}{c\tau+d} + \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{2ac|\tau|^2 + (ad+bc)(\tau+\bar{\tau}) + 2bd}{2|c\tau+d|^2} = \frac{(ad+bc)\text{Re }(\tau) + ac|\tau|^2 + bd}{|c\tau+d|^2}$$

$$\Rightarrow -\frac{\text{Re }\tau'}{\text{Im }\tau'} = -\frac{(ad+bc)\text{Re }(\tau) + ac|\tau|^2 + bd}{\text{Im }(\tau)} = -(ad+bc)\frac{\text{Re}(\tau)}{\text{Im }(\tau)} - ac\frac{|\tau|^2}{\text{Im }(\tau)} - bd\frac{1}{\text{Im }(\tau)}$$

$$\Rightarrow M'_{12} = (ad+bc)M_{12} - acM_{11} - bdM_{22}$$

This is consistent with our claim but the claim is not proven completely. We continue by calculating the transformations of the other components of $\mathcal{M}$. We see that;

$$|\tau'|^2 = \left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{a^2|\tau|^2 + 2ab\text{ Re }\tau + b^2}{|c\tau+d|^2} \Rightarrow \frac{|\tau'|^2}{\text{Im }\tau'} = \frac{a^2|\tau|^2 + 2ab\text{ Re }\tau + b^2}{\text{Im }\tau}$$

$$\Rightarrow M'_{11} = a^2 M_{11} - 2abM_{12} + b^2 M_{22}$$

Finally, we also see that;

$$M'_{22} = \frac{1}{\text{Im }\tau'} = \frac{|c\tau+d|^2}{\text{Im }\tau} = \frac{c^2|\tau|^2 + 2cd\text{ Re }\tau + d^2}{\text{Im }\tau} = c^2 M_{11} - 2cdM_{12} + d^2 M_{22}$$

Therefore, our claim was correct. Now, we can see that the following term is invariant;

$$\mathcal{M}_{ij}F_3^i F_3^j = F_3^i\mathcal{M}_{ij}F_3^j \to \Lambda^i{}_k F_3^k\left((\Lambda^{-1})^T\mathcal{M}\Lambda^{-1}\right)_{ij}\Lambda^j_l F_3^l = F_3^k\left(\Lambda^T(\Lambda^{-1})^T\mathcal{M}\Lambda^{-1}\Lambda\right)_{kl}F_3^l = F_3^k\mathcal{M}_{kl}F_3^l = \mathcal{M}_{kl}F_3^k F_3^l$$

Moreover, we see that;

$$\epsilon_{ij}\int C_4 \wedge F_3^i \wedge F_3^j \to \epsilon_{ij}\Lambda^i_l\Lambda^j_k\int C_4 \wedge F_3^l \wedge F_3^k = \epsilon_{lk}\det(\Lambda)\int C_4 \wedge F_3^l \wedge F_3^k = \epsilon_{lk}\int C_4 \wedge F_3^l \wedge F_3^k$$

So, we see that (12.1.15) is invariant under (12.1.16).

12.1.4 Type I superstring

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12.1.5 Heterotic strings

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12.2 Anomalies

12.2.1 Type II anomalies

12.2.2 Type I and heterotic anomalies

12.2.3 Relation to string theory

12.3 Superspace and superfields

A general coordinate transformation $z \to z'$ will transform the derivative ∂ as follows;

$$\partial = \frac{\partial z'}{\partial z}\partial' + \frac{\partial \bar{z}'}{\partial z}\bar{\partial}'$$

A conformal transformation is one for which ∂ is proportional to ∂'. Define superderivatives as follows;

$$D_\theta = \partial_\theta + \theta\partial_z, \quad \bar{D}_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$$

Then the general transformation for D_θ is as follows;

$$D_\theta = (\partial_\theta\theta'\partial_{\theta'} + \partial_\theta z'\partial_z + \partial_\theta\bar{\theta}'\partial_{\bar{\theta}'} + \partial_\theta\bar{z}'\partial_{\bar{z}'}) + \theta'(\partial\theta'\partial_{\theta'} + \partial z'\partial_z + \partial\bar{\theta}'\partial_{\bar{\theta}'} + \partial\bar{z}'\partial_{\bar{z}'})$$
$$= D_\theta\theta'\partial_{\theta'} + D_\theta z'\partial_{z'} + D_\theta\bar{\theta}'\partial_{\bar{\theta}'} + D_\theta\bar{z}'\partial_{\bar{z}'}$$

Now, we can say that the superconformal transformation is a transformation that sends D_θ to a multiple of itself. This implies that;

$$D_\theta\bar{\theta}' = D_\theta\bar{z}' = 0, \quad D_\theta z' = \theta' D_\theta\theta'$$

We now write z' and θ' in terms of z and θ. Now, we can expand both of them as a Taylor expansion in θ and it will give us two terms. Both of these terms will depend on z. These functions will be of different Grassmannian characters to preserve the Grassmannian character of the field. So, we have;

$$z'(z,\theta) = f(z) + \theta m(z), \ \theta'(z,\theta) = g(z) + \theta h(z)$$

where $f(z), h(z)$ are grassman even functions and $m(z), g(z)$ are grassman odd functions. We can constrain these fields by using the condition $D_\theta z' = \theta' D_\theta\theta'$. We proceed as follows;

$$\theta' D_\theta\theta' = (g(z) + \theta h(z))(h(z) + \theta\partial_z g(z)) = g(z)h(z) + \theta(h^2(z) - g(z)\partial_z g(z)), \ \ D_\theta z' = m(z) + \theta\partial_z f(z)$$
$$\Rightarrow m(z) = g(z)h(z), \ h(z) = \pm\sqrt{g(z)\partial_z g(z) + \partial_z f(z)}$$

Thus, we have the following;

$$z'(z,\theta) = f(z) + \theta g(z)h(z), \ \theta'(z,\theta) = g(z) + \theta h(z), \quad h(z) = \pm\sqrt{g(z)\partial_z g(z) + \partial_z f(z)} \tag{12.3.1}$$

For the infinitesimal transformations, we consider $f(z) = 1 + \epsilon v(z), \ g(z) = -i\epsilon\eta(z)$. This gives us;

$$h(z) = 1 + \frac{\epsilon}{2}\partial v(z), \ g(z)h(z) = -i\epsilon\eta(z), \ \Rightarrow \delta z = \epsilon(v(z) - i\theta\eta(z)), \ \delta\theta = \epsilon\left(-i\eta(z) + \frac{1}{2}\theta\partial v(z)\right)$$

(Check that it satisfies superconformal algebra) A tensor superfield of weight $(h, \tilde{h})$ transforms as follows;

$$(D_\theta\theta')^{2h}(D_{\bar{\theta}}\bar{\theta}')^{2\tilde{h}}\phi'(\mathbf{z},\mathbf{z}') = \phi(\mathbf{z},\mathbf{z})$$

where $\mathbf{z} = (z,\theta)$. For the transformation

$$\delta z = \epsilon\theta\eta(z), \quad \delta\theta = \epsilon\eta(z)$$

We see that;

$$D_\theta\theta' = 1 + \theta\epsilon\partial_z\eta, \ D_{\bar{\theta}}\bar{\theta}' = 1 + \bar{\theta}\epsilon\partial_{\bar{z}}\tilde{\eta}$$

and thus, we have;

$$(1 + \theta\epsilon\partial_z\eta)^{2h}\left(1 + \bar{\theta}\epsilon\partial_{\bar{z}}\tilde{\eta}\right)^{2\tilde{h}}\left(\phi'(\mathbf{z},\bar{\mathbf{z}}) + \theta\epsilon\eta\partial_z\phi' + \bar{\theta}\epsilon\tilde{\eta}\partial_{\bar{z}}\phi' + \epsilon\eta\partial_\theta\phi' + \epsilon\tilde{\eta}\partial_{\bar{\theta}}\phi'\right) = \phi(\mathbf{z},\bar{\mathbf{z}})$$

$$\Rightarrow \delta\phi(\mathbf{z},\bar{\mathbf{z}}) = -\epsilon\left(2h\theta\partial_z\eta + \eta Q_\theta + 2\tilde{h}\bar{\theta}\partial_{\bar{z}}\tilde{\eta} + \tilde{\eta}Q_{\bar{\theta}}\right)\phi'(\mathbf{z},\bar{\mathbf{z}}) = -\epsilon\left(2h\theta\partial_z\eta + \eta Q_\theta + 2\tilde{h}\bar{\theta}\partial_{\bar{z}}\tilde{\eta} + \tilde{\eta}Q_{\bar{\theta}}\right)\phi(\mathbf{z},\bar{\mathbf{z}}) + \mathcal{O}(\epsilon^2) \tag{12.3.2}$$

where I used binomial theorem and where $Q_\theta = \partial_\theta - \theta\partial_z, Q_{\bar{\theta}} = \partial_{\bar{\theta}} - \theta\partial_{\bar{z}}$. The holomorphic part of ϕ called $\phi(\mathbf{z})$ is expanded as follows;

$$\phi(\mathbf{z}) = \mathcal{O}(z) + \theta\Psi(z) \Rightarrow \delta\phi(\mathbf{z}) = \delta\mathcal{O}(z) + \theta\delta\Psi(z)$$

But from the result in (12.3.2), we see that;

$$\delta\phi(\mathbf{z}) = -\epsilon\left(2h\theta\partial_z\eta\phi(\mathbf{z}) + \eta\partial_\theta\phi(\mathbf{z}) - \eta\theta\partial_z\phi(\mathbf{z})\right) = -\epsilon\left(2h\theta\partial_z\eta\mathcal{O}(z) + \eta\Psi(z) - \eta\theta\partial_z\mathcal{O}(z)\right)$$
$$= -\epsilon\eta\Psi(z) - \epsilon\theta\left(2h\partial_z\eta\mathcal{O}(z) + \eta\partial_z\mathcal{O}(z)\right)$$

Comparing these two results for the variation of the field, we get;

$$\delta\mathcal{O} = -\epsilon\eta\Psi,\ \ \delta\Psi = -\epsilon\left(2h\partial_z\eta\mathcal{O}(z) + \eta\partial_z\mathcal{O}(z)\right)$$

A general superconfromal variation of a field $\mathcal{A}(z,\bar{z})$ is given as follows;

$$\delta\mathcal{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial^n\eta(z)G_{n-1/2} + (\partial^n\eta(z))^*\tilde{G}_{n-1/2}\right)\mathcal{A}(z,\bar{z}) \tag{12.3.3}$$

Comparing (12.3.3) with the variations above, we get;

$$G_{-1/2}\mathcal{O} = \Psi,\ G_r\mathcal{O} = 0\ \left(r \geq \frac{1}{2}\right)$$

$$G_{-1/2}\Psi = \partial_z\mathcal{O},\ G_{1/2}\Psi = 2h\mathcal{O},\ G_r\Psi = 0\ \left(r \geq \frac{3}{2}\right)$$

Now, we consider purely conformal transformations i.e.

$$z' = z + \epsilon v(z),\ \bar{z}' = \bar{z} + \epsilon\bar{v}(\bar{z}),\ \theta' = \theta + \frac{1}{2}\epsilon\theta\partial_z v(z),\ \bar{\theta}' = \bar{\theta} + \frac{1}{2}\epsilon\bar{\theta}\partial_{\bar{z}}\bar{v}(\bar{z})$$

Then, the variation in the tensor field is given as follows;

$$\delta\phi(\mathbf{z},\bar{\mathbf{z}}) = -\epsilon\left(h\partial_z v + \tilde{h}\partial_{\bar{z}}\bar{v} + v\partial_z + \bar{v}\partial_{\bar{z}} + \frac{1}{2}\theta\partial_z v\partial_\theta + \frac{1}{2}\bar{\theta}\partial_{\bar{z}}\bar{v}\partial_{\bar{\theta}}\right)\phi(\mathbf{z},\bar{\mathbf{z}}) + \mathcal{O}(\epsilon^2)$$

Now, again we consider the holomorphic part of $\phi(\mathbf{z},\bar{\mathbf{z}})$ and then, the variation becomes;

$$\delta\phi(\mathbf{z}) = -\epsilon\left(h\partial_z v\mathcal{O} + v\partial_z\mathcal{O}\right) - \epsilon\theta\left[\left(h + \frac{1}{2}\right)\partial_z v\Psi + v\partial_z\Psi\right]$$

$$\Rightarrow \delta\mathcal{O} = -\epsilon\left(h\partial_z v\mathcal{O} + v\partial_z\mathcal{O}\right),\ \ \delta\Psi = -\epsilon\left[\left(h + \frac{1}{2}\right)\partial_z v\Psi + v\partial_z\Psi\right]$$

A general conformal transformation of an operator is given as follows;

$$\delta\mathcal{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial^n v(z)L_{n-1} + (\partial^n v(z))^*\tilde{L}_{n-1}\right)\mathcal{A}(z,\bar{z}) \tag{12.3.4}$$

Comparing (12.3.4) with the variations above, we see that;

$$L_{-1}\mathcal{O} = \partial_z\mathcal{O},\ \ L_0\mathcal{O} = h\mathcal{O},\ \ L_n\mathcal{O} = 0\ \ (n > 0)$$

$$L_{-1}\Psi = \partial_z\Psi,\ \ L_0\Psi = \left(h + \frac{1}{2}\right)\Psi,\ \ L_n\Psi = 0\ \ (n > 0)$$

So, we see that $\mathcal{O}$ and Ψ have conformal dimensions h and $h + 1/2$. (**Show that the generator of world sheet rigid transformation is $G_{-1/2}$**).

12.3.1 Actions and backgrounds

We now calculate the superjacobian of the superconformal transformation given in (12.3.1) as follows;

$$\begin{vmatrix}\partial z'/\partial z & \partial z'/\partial\theta \\ \partial\theta'/\partial z & \partial\theta'/\partial\theta\end{vmatrix} = \begin{vmatrix}\partial_z f + \theta\partial_z(gh) & gh \\ \partial g + \theta\partial h & h\end{vmatrix} = (\partial_z f + g\partial_z g)(h + \theta\partial_z g) = h^2(z)D_\theta\theta'$$

(**Why this extra factor?**). The measure transforms as follows;

$$dz'\ d\theta' = D_\theta\theta'\ dz\ d\theta$$

The weight of this measure is $(-1/2, 0)$ because the weight of θ is $(1/2, 0)$. Including the measure $d\bar{z}d\bar{\theta}$, we see that the total measure has the weight $(-1/2, -1/2)$. Thus, the weight of the lagrangian density is $(1/2, 1/2)$. An example of such a lagrangian density is;

$$D_{\bar{\theta}}\mathbf{X}^\mu(z,\bar{z})D_\theta\mathbf{X}_\mu(z,\bar{z})$$

because the weight of $D_\theta \mathbf{X}^\mu(z,\bar{z})$ is $(1/2,0)$. Here, $\mathbf{X}^\mu(z,\bar{z})$ is a superfield. Its Taylor expansion can be done as follows;

$$\mathbf{X}^\mu(z,\bar{z}) = X^\mu(z,\bar{z}) + i\theta\psi^\mu(z) + i\bar{\theta}\bar{\psi}^\mu(z) + \theta\bar{\theta}F(z,\bar{z})$$

Here, the first three terms are easily understood. The fourth term is a possible term that can appear in Taylor's expansion but we will see that the F field is an auxiliary field. To write the action in terms of the components, we need the following small calculations;

$$\begin{aligned} D_\theta \mathbf{X}^\mu(\mathbf{z},\bar{\mathbf{z}}) &= (\partial_\theta + \theta\partial_z)\left(X^\mu(z,\bar{z}) + i\theta\psi^\mu(z) + i\bar{\theta}\bar{\psi}^\mu(z) + \theta\bar{\theta}F(z,\bar{z})\right) = \left(i\psi^\mu + \bar{\theta}F^\mu\right) + \theta\left(\partial_z X^\mu + i\bar{\theta}\partial_z\bar{\psi}^\mu\right) \\ &= i\psi^\mu + \theta\partial_z X^\mu + \bar{\theta}F^\mu + i\theta\bar{\theta}\partial_z\bar{\psi}^\mu \end{aligned} \tag{12.3.5}$$

Similarly,

$$\begin{aligned} D_{\bar{\theta}} \mathbf{X}^\mu(\mathbf{z},\bar{\mathbf{z}}) &= \left(\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}\right)\left(X^\mu(z,\bar{z}) + i\theta\psi^\mu(z) + i\bar{\theta}\bar{\psi}^\mu(z) + \theta\bar{\theta}F(z,\bar{z})\right) = (i\psi^\mu - \theta F^\mu) + \bar{\theta}\left(\partial_{\bar{z}} X^\mu + i\theta\partial_{\bar{z}}\psi^\mu\right) \\ &= i\bar{\psi}^\mu + \bar{\theta}\partial_{\bar{z}} X^\mu - \theta F^\mu - i\theta\bar{\theta}\partial_{\bar{z}}\psi^\mu \end{aligned} \tag{12.3.6}$$

Now, we calculate $D_{\bar{\theta}}\mathbf{X}^\mu D_\theta \mathbf{X}_\mu$ with retain only the terms that contain $\theta\bar{\theta}$ because the other terms vanish under $d^2\theta$ integration. This gives us;

$$D_{\bar{\theta}}\mathbf{X}^\mu(\mathbf{z},\bar{\mathbf{z}}) D_\theta \mathbf{X}_\mu(\mathbf{z},\bar{\mathbf{z}}) = \bar{\theta}\theta\left(\bar{\psi}^\mu\partial_z\bar{\psi}_\mu + \partial_z X^\mu \partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar{z}}\psi_\mu\right) + \text{ (non } \theta\bar{\theta} \text{ terms)} \tag{12.3.7}$$

and thus, we get (after doing the $d^2\theta$ integration);

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^2z d^2\theta \;\; \bar{\theta}\theta\left(\bar{\psi}^\mu\partial_z\bar{\psi}_\mu + \partial_z X^\mu \partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar{z}}\psi_\mu\right) \\ &= \frac{1}{4\pi}\int d^2z \left(\bar{\psi}^\mu\partial_z\bar{\psi}_\mu + \partial_z X^\mu \partial_z X_\mu + F^\mu F_\mu + \psi^\mu\partial_{\bar{z}}\psi_\mu\right) \end{aligned} \tag{12.3.8}$$

The vanishing of the variation of this action w.r.t F^μ gives;

$$\frac{1}{4\pi}\int d^2z F_\mu \delta F^\mu = 0 \Rightarrow F_\mu = 0 \tag{12.3.9}$$

The superfield action gives the following equation of motion (**derive this while being cautious about the minus signs**);

$$D_\theta D_{\bar{\theta}} X^\mu = 0 \tag{12.3.10}$$

To derive the OPE of the superfield that is invariant under rigid supersymmetry (i.e. when η is not dependent on z), we first derive the combinations that are invariant under rigid supersymmetry. The rigid supersymmetry is given as follows;

$$\delta z = -i\epsilon\theta\eta, \;\; \delta\theta = -i\epsilon\eta \tag{12.3.11}$$

Now, we calculate the transformation of the following quantities under supersymmetry;

$$\begin{aligned} z_1 - z_2 &\to (z_1 - i\epsilon\theta_1\eta) - (z_2 - i\epsilon\theta_2\eta) = z_1 - z_2 - i\epsilon(\theta_1 - \theta_2)\eta \\ \theta_1 - \theta_2 &\to (\theta_1 - i\epsilon\eta) - (\theta_2 - i\epsilon\eta) = \theta_1 - \theta_2 \end{aligned} \tag{12.3.12}$$

$$\begin{aligned} \theta_1\theta_2 &\to (\theta_1 - i\epsilon\eta)(\theta_2 - i\epsilon\eta) = \theta_1\theta_2 - i\epsilon(\theta_1 - \theta_2)\eta \\ &\Rightarrow z_1 - z_2 - \theta_1\theta_2 \to z_1 - z_2 - \theta_1\theta_2 \end{aligned} \tag{12.3.13}$$

So, we see that the quantities that are invariant under rigid supersymmetry are $\theta_1 - \theta_2$ and $z_1 - z_2 - \theta_1\theta_2$ (and their conjugates). So, the OPE can depend on these quantities only. The OPE can be calculated as follows (we set $\alpha' = 2$ in this section);

$$\begin{aligned} \mathbf{X}^\mu(\mathbf{z}_1,\bar{\mathbf{z}}_1)\mathbf{X}^\nu(\mathbf{z}_2,\bar{\mathbf{z}}_2) &= \left(X^\mu(z_1,\bar{z}_1) + i\theta_1\psi^\mu(z_1) + i\bar{\theta}_1\bar{\psi}^\mu(z_1)\right)\left(X^\mu(z_2,\bar{z}_2) + i\theta_2\psi^\mu(z_2) + i\bar{\theta}_2\bar{\psi}^\mu(z_2)\right) \\ &\sim -\eta^{\mu\nu}\ln|z_1 - z_2|^2 - \theta_1\theta_2\frac{\eta^{\mu\nu}}{z_1 - z_2} - \bar{\theta}_1\bar{\theta}_2\frac{\eta^{\mu\nu}}{\bar{z}_1 - \bar{z}_2} \end{aligned} \tag{12.3.14}$$

where we didn't write the auxiliary field. It can be written but that doesn't change the result (**Find more justification**). Now, we expand the following function as Taylor's expansion in θ_1 and $\bar{\theta}_1$;

$$\ln|(z_1-z_2-\theta_1\theta_2)|^2 = \ln\left[(z_1-z_2-\theta_1\theta_2)\left(\bar{z}_1-\bar{z}_2-\bar{\theta}_1\bar{\theta}_2\right)\right] = \ln|z_1-z_2|^2 + \frac{\theta_2}{z_1-z_2}\theta_1 + \frac{\bar{\theta}_2}{z_1-z_2}\bar{\theta}_1$$

$$= \ln|z_1-z_2|^2 + \frac{\theta_1\theta_2}{z_1-z_2} + \frac{\bar{\theta}_1\bar{\theta}_2}{z_1-z_2}0 \tag{12.3.15}$$

Using the above two equations, we have;

$$\mathbf{X}^\mu(\mathbf{z}_1,\bar{\mathbf{z}}_1)\mathbf{X}^\nu(\mathbf{z}_2,\bar{\mathbf{z}}_2) \sim -\eta^{\mu\nu}\ln|z_1-z_2-\theta_1\theta_2|^2 \tag{12.3.16}$$

We now write the ghost action in terms of two superfields i.e. B and C with weights $(\lambda-1/2,0)$ and $(1-\lambda,0)$. This means that the weight of $BD_{\bar{\theta}}C$ is

$$\left(\lambda-\frac{1}{2},0\right)+(1-\lambda,0)+\left(0,\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right) \tag{12.3.17}$$

Therefore, the following action has superconformal invariance;

$$S_{BC} = \frac{1}{2\pi}\int d^2z d^2\theta \;\; BD_{\bar{\theta}}C \tag{12.3.18}$$

The equations of motion are easily seen to be;

$$D_{\bar{\theta}}B = D_{\bar{\theta}}C = 0 \tag{12.3.19}$$

Now, we see that;

$$D_{\bar{\theta}}B = 0 \Rightarrow D_{\bar{\theta}}^2 B = 0 \Rightarrow \left(\partial_{\bar{\theta}}+\bar{\theta}\partial_{\bar{z}}\right)\left(\partial_{\bar{\theta}}+\bar{\theta}\partial_{\bar{z}}\right)B = \partial_{\bar{z}}B - \bar{\theta}\partial_{\bar{z}}\partial_{\bar{\theta}}B + \bar{\theta}\partial_{\bar{z}}\partial_{\bar{\theta}}B = \partial_{\bar{z}}B = 0 \Rightarrow \partial_{\bar{\theta}}B = 0 \tag{12.3.20}$$

The last equality follows because $D_{\bar{\theta}}B = \partial_{\bar{z}}B = 0$. Similarly, we can show that;

$$\partial_{\bar{\theta}}C = \partial_{\bar{z}}C = 0 \tag{12.3.21}$$

Therefore, the Taylor expansion of $B(\mathbf{z})$ and $C(\mathbf{z})$ are as follows;

$$B(\mathbf{z}) = \beta(z)+\theta b(z), \quad C(\mathbf{z}) = c(z)+\theta\gamma(z) \tag{12.3.22}$$

We see that the weights of all these components are exactly the weights that we find in bc ghost system and $\beta\gamma$ ghost system. The OPE is as follows;

$$B(\mathbf{z}_1)C(\mathbf{z}_2) = (\beta(z_1)+\theta_1 b(z_1))(c(z_2)+\theta_2\gamma(z_2)) = \theta_2\beta(z_1)\gamma(z_2)+\theta_1 b(z_1)c(z_2)$$

$$\sim -\theta_2\frac{1}{z_1-z_2}+\theta_1\frac{1}{z_1-z_2} = \frac{\theta_1-\theta_2}{z_1-z_2} \tag{12.3.23}$$

We can now do the following manipulation to show that this OPE is invariant under rigid supersymmetry;

$$\frac{\theta_1-\theta_2}{z_1-z_2-\theta_1\theta_2} = \frac{\theta_1-\theta_2}{z_1-z_2}\left(1-\frac{\theta_1-\theta_2}{z_1-z_2}\right)^{-1} = \frac{\theta_1-\theta_2}{z_1-z_2}\left(1-\frac{\theta_1+\theta_2}{z_1-z_2}\right) = \frac{\theta_1-\theta_2}{z_1-z_2} \tag{12.3.24}$$

Therefore, we have;

$$B(\mathbf{z}_1)C(\mathbf{z}_2) \sim \frac{\theta_1-\theta_2}{z_1-z_2-\theta_1\theta_2}$$

So we see that the OPE is written in terms of the quantities which are invariant under rigid supersymmetry. Now, we write down the sigma model action using superfields. We see that the following is true (**derive this**);

$$S = \frac{1}{4\pi}\int d^2z d^2\theta \left[G_{\mu\nu}(\mathbf{X})+B_{\mu\nu}(\mathbf{X})\right] D_{\bar{\theta}}\mathbf{X}^\nu D_\theta \mathbf{X}^\mu \tag{12.3.25}$$

$$= \frac{1}{4\pi}\int d^2z\left[\left(G_{\mu\nu}(X)+B_{\mu\nu}(X)\right)\partial_z X^\mu\partial_{\bar{z}}X_\mu + G_{\mu\nu}\left(\psi^\mu\mathcal{D}_{\bar{z}}\psi^\nu+\bar{\psi}^\mu\mathcal{D}_z\bar{\psi}^\nu\right)+\frac{1}{2}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\right] \tag{12.3.26}$$

where the covariant derivatives are defined as follows;

$$\mathcal{D}_{\bar{z}}\psi^\nu = \partial_{\bar{z}}\psi^\nu + \left[\Gamma^\nu_{\rho\sigma}(X) + \frac{1}{2}H^\nu_{\rho\sigma}(X)\right]\partial_{\bar{z}}X^\rho\psi^\sigma \tag{12.3.27}$$

$$\mathcal{D}_{z}\bar{\psi}^\nu = \partial_{z}\psi^\nu + \left[\Gamma^\nu_{\rho\sigma}(X) - \frac{1}{2}H^\nu_{\rho\sigma}(X)\right]\partial_{z}X^\rho\bar{\psi}^\sigma \tag{12.3.28}$$

(Talk about the R-R sector and the appearance of dilaton in the action).
In the heterotic string case, we have $\bar{\psi}^\mu$ in the left sector but no ψ^μ in the right sector. But there are λ^A fields in the left sector. So, the superfields required in this case are as follows;

$$\mathbf{X}^\mu(\mathbf{z},\bar{\mathbf{z}}) = X^\mu(z,\bar{z}) + i\bar{\theta}\bar{\psi}^\mu, \quad \boldsymbol{\lambda}^A(\mathbf{z}) = \lambda^A + \theta G^A$$

(Write more about the heterotic case and anomalies)

12.3.2 Vertex operators

A naive idea of pictures is as follows. Positions of vertex operators in bosonic strings can be fixed but they come with an additional $c(z)\tilde{c}(\bar{z})$ factor. The positions of the vertex operators can be unfixed and they will have an d^2z integral with them. In the superstring case, we call these different categories of vertex operators to be in different **pictures**. The analogue of $c(z)\tilde{c}(\bar{z})$ in the NS sector case is (**find more about this**);

$$e^{-\phi(z)-\tilde{\phi}(\bar{z})}$$

which is the NS vacuum vertex operator. In the superstring case, we integrate over $(\theta,\bar{\theta})$ in addition to $(z,\bar{z})$. Integrating over $d^2\theta$ takes out the $\theta\bar{\theta}$ component. Let's call this component Ψ. This component can also be taken out by applying $G_{-1/2}\tilde{G}_{-1/2}$ on the lowest weight component (which we can refer to as $\mathcal{O}$). So, we have;

$$G_{-1/2}\tilde{G}_{-1/2}\mathcal{O} = \Psi$$

So, the $d^2\theta$ integration can be replaced with $G_{-1/2}\tilde{G}_{-1/2}$ acting on lowest weight components of the superfield. However, in the unfixed operators, we won't have $e^{-\phi(z)-\tilde{\phi}(\bar{z})}$ factor, just like the bosonic case where $c(z)\tilde{c}(\bar{z})$ is absent from the vertex operators whose position aren't fixed. So, we see that the $(\phi,\tilde{\phi})$ charges of the unfixed operators is $(0,0)$ and it is $(-1,-1)$ for the fixed operators. This fact is referred to by saying that unfixed operators are in the $(0,0)$ picture and the fixed operators are in the $(-1,-1)$ picture. In other words, having $(\phi,\tilde{\phi})$ charge equal to $(q,\tilde{q})$ is defined as being in the $(q,\tilde{q})$ picture.
As an example, we derive the $(0,0)$ and $(-1,-1)$ picture vertex operators for the following massless states in the NS – NS sector;

$$\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|0;k\rangle_{\text{NS}} \tag{12.3.29}$$

The unfixed vertex operators for these states are as follows;

$$\mathcal{V}^{-1,-1} = g_c e^{ik.X}e^{-\phi-\tilde{\phi}}\psi^\mu\tilde{\psi}^\nu$$

where g_c is the closed string coupling constant. To find the vertex operators in the $(0,0)$ picture, we apply the $G_{-1/2}\tilde{G}_{-1/2}$ operator on this vertex operators and get rid of the $e^{-\phi-\tilde{\phi}}$ factor. For that, we will need the following;

$$G_r = \sum_{n\in\mathbb{Z}}\alpha^\mu_n\psi_{\mu r-n} \Rightarrow G_{-1/2} = \sum_{n\in\mathbb{Z}}\alpha^\mu_n\psi_{\mu,-1/2-n} = \alpha^\mu_0\psi_{\mu,-1/2} + \sum_{n=1}^{\infty}\left(\alpha^\mu_n\psi_{\mu,-1/2-n} + \alpha^\mu_{-n}\psi_{\mu,-1/2+n}\right) \tag{12.3.30}$$

where the commas have been placed to differentiate between Lorentz indices and Fourier indices. For the state in (12.3.29), no term in the first term in the sum in (12.3.30) contributes because all the α's in that term are annihilation operators. From the second term in the sum, only $n=1$ term contributes because all the other ψ modes annihilate the state in (12.3.29). Similar arguments also apply for $\tilde{G}_{-1/2}$ Lastly, the zero mode term in $G_{-1/2}$ also contributes. So, effectively, we have;

$$G_{-1/2}\tilde{G}_{-1/2}\psi^\alpha_{-1/2}\tilde{\psi}^\beta_{-1/2}|0;k\rangle_{\text{NS}} = \left(\alpha^\mu_0\psi_{\mu,-1/2} + \alpha^\mu_{-1}\psi_{\mu,1/2}\right)\left(\tilde{\alpha}^\nu_0\tilde{\psi}_{\nu,-1/2} + \tilde{\alpha}^\nu_{-1}\tilde{\psi}_{\nu,1/2}\right)\psi^\alpha_{-1/2}\tilde{\psi}^\beta_{-1/2}|0;k\rangle_{\text{NS}}$$

Now, using the anticommutation relations for the fermionic modes, we have;

$$-\left(\alpha_0^\mu \psi_{\mu,-1/2}\psi^\alpha_{-1/2} + \alpha^\mu_{-1}\delta^\alpha_\mu\right)\left(\tilde{\alpha}^\nu_0\tilde{\psi}_{\nu,-1/2}\tilde{\psi}^\beta_{-1/2} + \tilde{\alpha}^\nu_{-1}\delta^\beta_\nu\right)|0;k\rangle_{\rm NS}$$

$$= -\left(\alpha_0.\psi_{-1/2}\psi^\alpha_{-1/2} + \alpha^\alpha_{-1}\right)\left(\tilde{\alpha}_0.\tilde{\psi}_{-1/2}\tilde{\psi}^\beta_{-1/2} + \tilde{\alpha}^\beta_{-1}\right)|0;k\rangle_{\rm NS} \qquad (12.3.31)$$

where the overall minus sign comes because ψ^α travels through the $\tilde{\psi}$ bracket and both of the terms in the bracket contain a single spinor mode. Now, the vertex operators corresponding to these states are as follows;

$$\mathcal{V}^{0,0} = -g_c\left(k.\psi\psi^\alpha + i\partial_z X^\alpha\right)\left(k.\tilde{\psi}\tilde{\psi}^\beta + i\partial_{\bar{z}}X^\beta\right)e^{ik.X}$$

This vertex operator is in the $(0,0)$ picture because there is no $(\phi,\tilde{\phi})$ charge. This expression is in $\alpha' = 2$ convention. Similarly for open strings, we have the following massless state;

$$\psi^\mu_{-1/2}|0;k;a\rangle_{\rm NS}$$

where a is the Chan Paton index. Now, the vertex operator in the -1 picture corresponding to this state is;

$$\mathcal{V}^{-1} = g_o e^{-\phi}\psi^\mu t^a e^{ik.X} \qquad (12.3.32)$$

with t^a being the gauge group generator and with $d^2zd^2\theta$ integrations suppressed. Now, the picture 0 vertex operator is derived just like the calculation above. We get;

$$G_{-1/2}\psi^\mu|0;k;a\rangle = \left(\alpha_0.\psi_{-1/2}\psi^\mu_{-1/2} + \alpha^\mu_{-1}\right)|0;k;a\rangle_{\rm NS} \Rightarrow \mathcal{V}^0 = g_o t^a(\partial_z X^\mu + 2k.\psi\psi^\mu)$$

$$= g_0(2\alpha')^{-1/2}t^a(i\dot{X}^\mu + 2\alpha' k.\psi\psi^\mu)e^{ik.X} \qquad (12.3.33)$$

(**Restore α' factors and the issue of the dot derivative. Talk about heterotic vertex operators and couplings**).

12.4 Tree level amplitudes

In this section, we quote and motivate some of the results that will be derived carefully later. While studying bosonic theory on the sphere, we fixed the $(z,\bar{z})$ positions of three vertex operators and got factors of $c(z)\tilde{c}(\bar{z})$. The positions of three vertex operators are fixed because there are three complex killing vectors on the sphere.

In the superstring theory, we need to determine the number of vertex operators whose $(\theta,\bar{\theta})$ coordinates can be fixed. To know the answer to this, we need to study superconformal killing vectors. Polchinski casts these results in terms of the zero modes of c and γ fields. However, we can just quote here that $(\theta,\bar{\theta})$ coordinates of two vertex operators should be fixed on the sphere and we get factors of $e^{-\phi-\tilde{\phi}}$ whenever we fix one operator. In other words, we need two operators in $(-1,-1)$ picture, and all other operators are in $(0,0)$ picture. (**Talk about the ϕ anomaly argument**). For open strings on the disc, we similarly need two vertex operators in the -1 picture and the rest in 0 picture.

The vertex operator of the R ground state is

$$\mathcal{V}_{\mathbf{s}} = e^{-\phi/2}\Theta_{\mathbf{s}} \qquad (12.4.1)$$

and thus, it has a ϕ charge of $-1/2$. Due to the conservation of ϕ charge (**where does this come from?**), we know that the total ϕ charge is -2. We just quote the interpretation here and leave the justification for later. The ϕ charge for the fermions is $-1/2$ such that the total ϕ charge is -2. In the case of two fermions, we have ϕ charge equal to $-1/2$ for fermions and -1 for bosons.

12.4.1 Three point amplitudes

Type I disc amplitude: Type I disc amplitude for three bosons is as follows;

$$\frac{1}{\alpha' g_o^2}\langle c\mathcal{V}^{-1}(x_1)c\mathcal{V}^{-1}(x_2)c\mathcal{V}^0(x_3)\rangle + (1\leftrightarrow 2) \qquad (12.4.2)$$

where we fixed the $(\theta, \bar{t})$ coordinates of two operators and the $(z, \bar{z})$ coordinates of all three of them. The overall normalization is taken to be the same as C_{D^2} in (6.5.4). This normalization was derived using unitarity. The $1 \leftrightarrow 2$ term comes due to summing over the other ordering. For calculating this amplitude, we need the following expectation values;

$$\langle c(x_1)c(x_2)c(x_3)\rangle = x_{12}x_{23}x_{31}, \quad \langle e^{-\phi/2}(x_1)e^{-\phi/2}(x_2)\rangle = x_{12}^{-1} \tag{12.4.3}$$

The c expectation value was calculated in chapter 6 while the ϕ expectation value was calculated in (10.4.13). We have chosen a normalization for this OPE. Using (12.3.32) and (12.3.33), we see that we also need the following expectation value;

$$\left\langle \psi^\mu e^{ik_1.X}(x_1)\psi^\nu e^{ik_2.X}(x_2)\left(i\dot{X}^\rho + 2\alpha' k.\psi\psi^\rho e^{ik.X}(x_3)\right)\right\rangle$$

This expectation value is as follows (**derive this**);

$$\left\langle \psi^\mu e^{ik_1.X}(x_1)\psi^\nu e^{ik_2.X}(x_2)\left(i\dot{X}^\rho + 2\alpha' k.\psi\psi^\rho e^{ik.X}(x_3)\right)\right\rangle = -2i\alpha'(2\pi)^{10}\delta\left(\sum_i k_i\right)\left(\frac{\eta^{\mu\nu}k_1^\rho}{x_{12}x_{13}} + \frac{\eta^{\mu\nu}k_2^\rho}{x_{12}x_{23}} + \frac{\eta^{\nu\rho}k_3^\mu - \eta^{\mu\rho}k_3^\nu}{x_{13}x_{23}}\right)$$

12.4.2 Four point amplitudes

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12.5 General amplitudes

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12.5.1 Pictures

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12.5.2 Super-Riemann surfaces

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12.5.3 The measure on supermoduli space

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12.6 One-loop amplitudes

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12.6.1 Non-renormalization theorems

13 Chapter 13: D Branes

We do know that under T duality (in the 9−th direction say), we have to do the following change;

$$X'^9_{\rm R}(\bar z) = -X^9_{\rm R}(\bar z)$$

This implies sending all the Fourier components of $X_{\rm R}$ to negative of themselves. This will send $\tilde T_{\rm F}$ to negative of itself and it won't change any other stress tensor. This will disrupt the superconformal algebra. Since $\tilde T_{\rm F}$ contains $\tilde\psi^\mu \partial X^\mu_{\rm R}$, we can also do the following change;

$$\tilde\psi'^9(\bar z) = -\psi^9(\bar z)$$

to save the superconformal algebra. Using the definition of $\tilde F$, we see that $\tilde F$ contains $\tilde S_4$ and $\tilde S_4$ contains $\tilde\psi^9_r$. Thus, the value of $\tilde S_4$ changes on its eigenstates by its negative. Since the possible values of $\tilde S_4$ are ±1/2, $\tilde S_4$ going negative to itself means that there is a difference of ±1 and thus, there is a difference of ±1 in $\tilde F$. Hence, $e^{\pi i \tilde F}$ (which is just chirality on the ground state) gets reversed on the Ramond right-handed ground state. So, we see that type IIA and type IIB are T duals of each other (where T duality is done on an odd number of dimensions). Now, we relate R − R fields in T dual theories. Consider the vertex operators of R-R ground states;

$$\psi_0^{[\mu_1}...\psi_0^{\mu_p]}|\alpha;k\rangle \propto \Gamma^{[\mu_1}...\Gamma^{\mu_p]}\mathcal{V}_\alpha|1\rangle$$

So, we need to ask that how do $\mathcal{V}_\alpha$ transform under T duality in 9 direction. Since $\mathcal{V}_\alpha$ has a spinor index, it should transform like a spinor under parity in the 9 direction. The corresponding operator is $\beta^9_{\alpha\beta}$. This operator anti commutes with Γ^9 but commutes with all the other Γ's (**include justification**). So, we have an obvious solution of the following;

$$\beta^9 = \Gamma^9\Gamma \Rightarrow \beta^9\Gamma^9 = \Gamma^9\Gamma\Gamma^9 = -\Gamma^9\beta^9,\ \beta^9\Gamma^\mu = \Gamma^\mu\beta^9 \quad \mu \neq 9$$

So, we now see that under T duality, if $\mu = 9$ is present in the R vertex operator, then it gets absorbed by the extra Γ^9 coming from the transformation of $\mathcal{V}$ and if there is no $\mu = 9$ present in the vertex operator, then it get a $\mu = 9$ index. This transforms between type IIA and type IIB R − R fields. For more than one T dualized direction, the parity operators get multiplied;

$$\beta = \prod_m \beta^m$$

We derive a couple of identities now. Firstly, we see that;

$$\beta^m\beta^n = \Gamma^m\Gamma\Gamma^n\Gamma = \Gamma^n\Gamma^m\Gamma\Gamma = -\Gamma^n\Gamma\Gamma^m\Gamma = -\beta^n\beta^m, \quad (m \neq n)$$

So, T dualities in different directions don't commute. (**derive the presence of fermion number there**).

13.1 T duality of type II strings

tt

13.2 T duality of type I string

(Include the supersymmetry charges thing and the worldsheet supercurrent thing). D-p branes naturally couple to $p+1$ form fields. They can couple to $R-R$ fields;

$$\int C_{p+1}$$

(Do they couple to the B field as well i.e. D1 brane?). Type II A has 1, 3, 5, 7 and 9 form potentials and thus, it has D-p branes with p even. Type II B has 0, 2, 4, 6, 8 and 10 form fields and thus, it has D-p branes with p odd.

13.2.1 New connections between string theories

The susy algebra for extended objects is as follows (**find justification for this**);

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\left[P_M + \frac{Q_M^{\rm NS}}{2\pi\alpha'}\right]\Gamma^M_{\alpha\beta} \Rightarrow \{Q_\alpha, Q_\beta^\dagger\} = 2\left[P_M + \frac{Q_M^{\rm NS}}{2\pi\alpha'}\right](\Gamma^0\Gamma^M)_{\alpha\beta} \tag{13.2.1}$$

$$\{\tilde{Q}_\alpha, \bar{\tilde{Q}}_\beta\} = -2\left[P_M - \frac{Q_M^{\rm NS}}{2\pi\alpha'}\right]\Gamma^M_{\alpha\beta} \Rightarrow \{\tilde{Q}_\alpha, \tilde{Q}_\beta^\dagger\} = 2\left[P_M - \frac{Q_M^{\rm NS}}{2\pi\alpha'}\right](\Gamma^0\Gamma^M)_{\alpha\beta} \tag{13.2.2}$$

$$\{Q_\alpha, \bar{\tilde{Q}}_\beta\} = -2\sum_p \frac{\tau_p}{p!} Q^{\rm R}_{M_1\ldots M_p}\left(\beta^{M_1}\ldots\beta^{M_p}\right)_{\alpha\beta} \Rightarrow \{Q_\alpha, \tilde{Q}_\beta^\dagger\} = 2\sum_p \frac{\tau_p}{p!} Q^{\rm R}_{M_1\ldots M_p}\left(\beta^{M_1}\ldots\beta^{M_p}\Gamma^0\right)_{\alpha\beta} \tag{13.2.3}$$

where $Q^{\rm NS}$ and $Q^{\rm R}$ are NS and R charges. Recall that charges carried by extended objects have multiple indices.

(Include the arguments from type I divergence, Include the arguments for the supersymmetry algebra, D instanton example)

13.3 D Branes action and charges

We need to calculate the D brane tension in superstrings. For that, we need to calculate the cylinder amplitude and extract the massless divergence from it for an open string stretched between two branes. There are contributions from NS and R sectors. The NS amplitude was already calculated in section 10.8 and the only change is the integration happening on a different number of dimensions and the extra factor due to stretched string between two branes. The intervening steps are exactly like the steps in section 8.7 while calculating the D-brane tension. We get;

$$\mathcal{A}_{\rm NS-NS} = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty \frac{dt}{t^{p+3/2}} e^{-ty^2/2\pi\alpha'} t^4 = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(5-p)/2} e^{-ty^2/2\pi\alpha'}$$

$$= \frac{8^2 iV_{p+1}}{8\pi(8\pi^2\alpha')^5}\int_0^\infty \pi\frac{dt}{t^2}(8\pi^2\alpha' t)^{(9-p)/2} e^{-ty^2/2\pi\alpha'} = \frac{4\times 16i\, V_{p+1}}{8\pi(8\pi^2\alpha')^5}\int_0^\infty \pi\frac{dt}{t^2}(8\pi^2\alpha' t)^{(9-p)/2} e^{-ty^2/2\pi\alpha'}$$

The steps in the last line are done to bring the expression into the standard form (that resembles other expressions that we derived before). Now, we do this integral but for that, we use a simplified form of this massless divergence. So, we have;

$$\mathcal{A}_{\rm NS-NS} = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(5-p)/2} e^{-ty^2/2\pi\alpha'} = \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{y^2}\right)^{(7-p)/2}\int_0^\infty d\mu \mu^{(5-p)/2} e^{-\mu}$$

$$= \frac{8iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{y^2}\right)^{(7-p)/2}\Gamma\left(\frac{7-p}{2}\right)$$

Now, we use (8.7.6) with $d = 9-p$ to get;

$$G_{9-p}(y) = \frac{1}{4\pi^{(9-p)/2}|y|^{7-p}}\Gamma\left(\frac{7-p}{2}\right) \Rightarrow \Gamma\left(\frac{7-p}{2}\right) = 4\pi^{(9-p)/2}|y|^{7-p} G_{9-p}(y)$$

Putting this equation into the expression of $\mathcal{A}_{\rm NS-NS}$, we get;

$$\mathcal{A}_{\rm NS-NS} = 2\pi i V_{p+1}(4\pi^2\alpha')^{3-p} G_{9-p}(y) \tag{13.3.1}$$

Now, for the field theory calculation, the expectation values become the following by putting $D = 10$ in (8.7.10) and (8.7.11);

$$\langle\tilde{\phi}\tilde{\phi}\rangle = -\frac{2i\kappa^2}{k^2}$$

(Do the graviton expectation and the field theory calculation). The field theory result is given as follows;

$$\frac{2i\kappa^2\tau_p^2}{k_\perp^2}V_{p+1} = 2i\kappa^2\tau_p^2 V_{p+1} G_{9-p}(y) \tag{13.3.2}$$

Equality of (13.3.1) and (13.3.2), we get;

$$\tau_p^2 = \frac{\pi(4\pi^2\alpha')^{3-p}}{\kappa^2} \Rightarrow \tau_{p-1}^2 = \frac{\pi(4\pi^2\alpha')^{3-(p-1)}}{\kappa^2} = \frac{\pi(4\pi^2\alpha')^{3-p}}{\kappa^2}4\pi\alpha' = 4\pi\alpha'\tau_p^2 \Rightarrow \tau_{p-1} = 2\pi\sqrt{\alpha'}\,\tau_p \Rightarrow T_{p-1} = 2\pi\sqrt{\alpha'}\,T_p$$

So, we see that the recursion relation (8.7.5) is still satisfied (recall that τ_p and T_p are related linearly by the exponential of dilaton expectation value). To guess the low energy R – R exchange action is easy. We have two $D-p$ branes (with couple to C_{p+1} field and the action for this field is);

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}|F_{p+2}|^2$$

Moreover, the branes are sources for C_{p+1} fields and thus, the action of the coupling of branes to C_{p+1} field is (with coupling constant μ_p);

$$\mu_p\int C_{p+1}$$

So, the total action is;

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}|F_{p+2}|^2 + \mu_p\int C_{p+1}$$

The propagator for the C_{p+1} field is easy to read as the normalization is canonical. One can redefine a new C_{p+1} field as $(\sqrt{2}\kappa_{10})^{-1}C_{p+1}$ and then, the normalization and propagator are the standard photon normalization and propagator i.e. i/k^2. Be careful about the different signature of the metric used usually in QFT. This is the reason there is no minus sign in the propagator here. The propagator is as follows;

$$2\kappa_{10}^2(\text{standard propagator}) = \frac{2i\kappa_{10}^2}{k^2}$$

The field theory amplitude is (**derive this and there should be a factor of Vp+1**);

$$-2i\kappa_{10}^2\mu_p^2G_{9-p}(y)V_{p+1}$$

Since this should be negative of the NS – NS field theory result, using (13.3.1) we have;

$$-2i\kappa_{10}^2\mu_p^2G_{9-p}(y)V_{p+1} = -2\pi iV_{p+1}(4\pi^2\alpha')^{3-p}2G_{9-p}(y) \Rightarrow \mu_p^2 = \frac{\pi}{\kappa_{10}^2}(4\pi\alpha')^{3-p} = \frac{\kappa^2}{\kappa_{10}^2}\tau_p^2 = e^{2\phi_0}\tau_p^2 = T_p^2 \quad (13.3.3)$$

(Find justification of second last step. Probably from chapter 12).**(Write about orientifolds and pictures)**.

13.3.1 Dirac quantization condition

We first review the Dirac condition for normal electrodynamics. Since the electrical charge is carried by point particles (or 0 branes), the field strength should be F_2 and since normal electrodynamics is in four dimensions, the dual field strength is also a two-form which in turn implies that magnetic sources should also be 0 branes or point particles.
Consider a magnetic monopole with magnetic charge μ_m and a sphere S^2 around it. The flux through S^2 is calculated as follows;

$$\mu_m = \int_{S^2} B_i dA_i = \int_{S^2}\frac{1}{2}\epsilon_{ijk}F_{jk}dA_i \quad (13.3.4)$$

Now, we do the following manipulation;

$$dA_i = (dx^1\times dx^2)_i = \epsilon_{ilm}dx_l^1dx_m^2$$

$$\Rightarrow \epsilon_{ijk}dA_i = \epsilon_{ijk}\epsilon_{ilm}dx_l^1dx_m^2 = (\delta_{lj}\delta_{mk}-\delta_{lk}\delta_{mj})dx_l^1dx_m^2 = dx_j^1dx_k^2 - dx_k^1dx_j^2 = 2dx_j^1\wedge dx_k^2 \quad (13.3.5)$$

Using (13.3.5) in (13.3.4), we get;

$$\int_{S^2}\epsilon_{ijk}F_{jk}dx_j^1\wedge dx_k^2 = \int_{S^2}F_2 = \mu_m \quad (13.3.6)$$

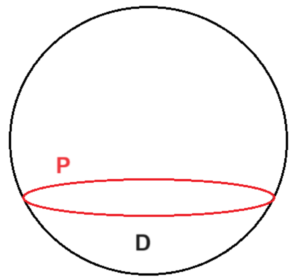


Figure 1: The path on which integration is performed

Including an electrically charged particle in this field with charge μ_e, we get the following phase (**include an explanation**) (the path is shown in the figure below);

$$\exp\left(i\mu_e \oint_P A_1\right) = \exp\left(i\mu_e \oint_D F_2\right) = \exp\left(i\mu_e\mu_m\right)$$

To ensure the single-valuedness of this quantity, we need to ensure that;

$$\mu_e\mu_m = 2\pi n$$

Now, we generalize this discussion to $p-$branes. If the source is a $p-$brane, then the field strength is F_{p+2}, the dual strength is $F_{10-p-2} = F_{8-p}$ and thus, the source of such a field is a $8-p-2 = (6-p)$ brane. We call this brane a magnetic brane and p brane an electric brane. Now, in 3 space dimensions, a point is surrounded by S^2 (two dimensions), and a line is surrounded by S^1 (one dimension). The dimension of the sphere S^k surrounding an extended entity of p dimensions in a d dimensional space is given by the formula;

$$k = d - p - 1 \tag{13.3.7}$$

which reproduces the results quoted above. Now, for 9 space dimensions, a $6-p$ magnetic brane is surrounded by a sphere of dimension

$$9 - (6-p) - 1 = p + 2$$

which makes sense as it is the number of indices on F_{p+2} and we expected the generalization of (13.3.6) to be something like F_{p+2} integrated over S^{p+2}. So, the generalization of (13.3.6) is confirmed to be (**find justification for the factor**);

$$\int_{S^{p+2}} F_{p+2} = 2\kappa_{10}^2\mu_{6-p}$$

An argument like before gives us the following condition;

$$\exp\left(i\mu_p \oint_{S^{p+2}} F_{p+2}\right) = \exp\left(2i\kappa_{10}^2\mu_p\mu_{6-p}\right) \text{is single-valued} \Rightarrow 2\kappa_{10}^2\mu_p\mu_{6-p} = 2\pi n \Rightarrow \mu_p\mu_{6-p} = \frac{\pi n}{\kappa_{10}^2} \tag{13.3.8}$$

Using (13.3.3) we get the following;

$$\mu_p = \frac{\sqrt{\pi}}{\kappa_{10}}(4\pi\alpha')^{(3-p)/2}, \;\; \mu_{6-p} = \frac{\sqrt{\pi}}{\kappa_{10}}(4\pi\alpha')^{(p-3)/2} \Rightarrow \mu_p\mu_{6-p} = \frac{\pi}{\kappa_{10}^2}$$

So, the charges in (13.3.3) satisfy (13.3.8) for $n = 1$.

13.3.2 D-brane actions

The massless (NS+, NS+) spectrum is the same as the bosonic massless spectrum, and the arguments from section 8.7 apply. So, the coupling of the D-brane with NS – NS sector fields is like the DBI action. We replace T_p with μ_p because they are equal as shown above and to include the case of multiple branes, we include a trace because the fields become gauge fields. So, the NS – NS action becomes;

$$S_{\text{NS-NS}} = -\mu_P \int d^{p+1}\xi \;\text{Tr}\left[-e^{-\phi}\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)\right] \tag{13.3.9}$$

(Include the discussion about flat directions). The coupling to R−R fields is derived as follows (**derive this**);

$$S_{\text{R-R}} = i\mu_p \int_{p+1} \text{Tr}\left[\exp\left[2\pi\alpha' F_2 + B_2\right] \wedge \sum_q C_q\right] \tag{13.3.10}$$

(Find about spacetime curvature couplings, the fermionic part of the action and the full non-linear action). To find the YM coupling on the brane,

13.3.3 Coupling constants

We now calculate the ratio τ_{F1} to τ_{D1} i.e. the ration of tension of fundamental string to the tension in $D1$ brane. It is as follows;

$$\frac{\tau_{F1}}{\tau_{D1}} = \frac{e^{-\phi_0}/2\pi\alpha'}{e^{-\phi_0}\sqrt{\pi}(4\pi^2\alpha')/\kappa} = \frac{\kappa}{8\pi^{7/2}\alpha'^2}$$

Now, as shown in chapter 6 (**do it in chapter 12 too**), g_c is proportional to κ and thus;

$$g \propto \frac{\tau_{F1}}{\tau_{D1}}$$

Although we fixed the normalization between g_o and g_c in chapter 8 (**do it for superstrings as well**), we didn't fix the normalization of $g_c \propto e^{\phi}$. So, we define g_c to be the ratio above;

$$g_c = \frac{\tau_{F1}}{\tau_{D1}}$$

Then, using the expression for τ_{F1}/τ_{D1}, we get;

$$\kappa = 8g_c\pi^{7/2}\alpha'^2 \Rightarrow \kappa^2 = \frac{1}{2}2^7 g_c^2\pi^7\alpha'^4 = \frac{1}{2}g_c^2(2\pi)^7\alpha'^4$$

The string tension can then be written in terms of g_c as follows;

$$\tau_p = \frac{\sqrt{\pi}}{\kappa}(4\pi\alpha')^{(3-p)/2} = \frac{\sqrt{\pi}(4\pi\alpha')^{(3-p)/2}\sqrt{2}}{g_c(2\pi)^{7/2}\alpha'^2} = (2\pi)^{-p}\alpha'^{-(1+p)/2}g_c^{-1} \tag{13.3.11}$$

Notice that this is dimensionless only if $p = -1$ (for instanton) **(Find more about this)**. To derive the Yang-Mills coupling on the D-brane, we expand (13.3.9) for $G_{ab} = \eta_{ab}, B_{ab} = 0$ and $\phi = \phi_0$. We also use the following identity;

$$\det(-\mathbb{I}+M) = -1+\text{Tr}(M)+\mathcal{O}(M^2) \Rightarrow \det(\eta_{ab}+2\pi\alpha' F_{ab}) = -1+2\pi\alpha' F_{ab}F^{ba}+\mathcal{O}(F^2) \sim -1-2\pi\alpha' F_{ab}F^{ab} = -1-2\pi\alpha' F^2$$

So, the expansion of (13.3.9) is as follows;

$$-\mu_p \int d^{p+1}\xi\, \text{Tr}\left[e^{-\phi_0}(1+2\pi\alpha' F^2)\right] = -\mu_p \int d^{p+1}\xi\, \text{Tr}\left[e^{-\phi_0}\right] - 2\pi\alpha'\mu_p \int d^{p+1}\xi\, \text{Tr}\left[e^{-\phi_0}F^2\right]$$

(Trace on the dilaton is giving a problem. If we ignore the dilaton trace, then it can be taken out and mu becomes tau then. The action then becomes);

$$-2\pi\alpha'\tau_p \int d^{p+1}\xi\, \text{Tr}\left[F^2\right] = \frac{8\pi\alpha'\tau_p}{-4} \int d^{p+1}\xi\, \text{Tr}\left[F^2\right] = -\frac{1}{4}\frac{(2\pi\alpha')^2}{\pi\alpha'/2}\tau_p \int d^{p+1}\xi\, \text{Tr}\left[F^2\right]$$

(The extra factor of $\pi\alpha'/2$ is coming in denominator. How to deal with this?). Now, if we read g_{YM}^2, we get;

$$\frac{1}{g_{YM}^2} = (2\pi\alpha')^2\tau_p \Rightarrow g_{YM}^2 = \frac{1}{(2\pi\alpha')^2\tau_p} = (2\pi)^{-2}\alpha'^{-2}\tau_p^{-1} = (2\pi)^{-2}\alpha'^{-2}(2\pi)^p\alpha'^{(1+p)/2}g_c = (2\pi)^{p-2}\alpha'^{(p-3)/2}g_c$$

We see that this coupling is dimensionless only for $p = 3$ i.e. three-dimensional space case which we already know is true.
(The relation between g_{YM}, κ and α' requires some orientifold based reasoning. Do that.), The result is;

$$\frac{g_{YM}^2}{\kappa} = 2(2\pi)^{7/2}\alpha' \quad \text{(Type I)}$$

(Talk about the Born-Infeld action for type-I)

13.4 D-brane interactions: statics

We consider two branes of possibly different dimensions i.e. a $D-p$ brane and a $D-p'$ brane. Let's call the set of coordinates that have Dirichlet boundary conditions on the first and second brane be S_D and S'_D respectively. The corresponding sets for Neumann directions are S_N and S'_N. Then the DD directions are $S_D \cap S'_D$ and so on. The unbroken supersymmetries due to the first and second D brane are as follows;

$$Q_\alpha + (\beta^\perp Q)_\alpha, \quad Q_\alpha + (\beta'^\perp Q)_\alpha = Q_\alpha + (\beta^\perp \beta^{\perp -1} \beta'^\perp Q)_\alpha \tag{13.4.1}$$

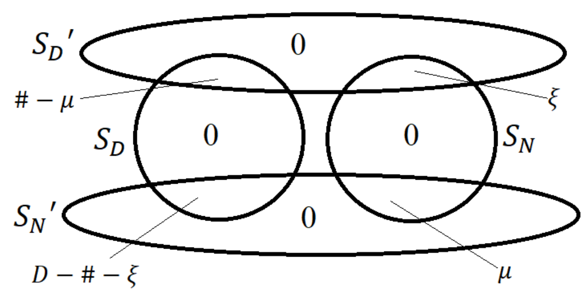
and thus, the supersymmetries unbroken by both D-branes are the ones which are generated by Q_α such that;

$$\beta^{\perp -1} \beta'^\perp Q_\alpha = Q_\alpha \tag{13.4.2}$$

because if this happens, then the spinors unbroken by the second one and the first one are the same as can be seen from (13.4.1). So, we need to find the number of solutions of (13.4.2). Now, $\beta^{\perp -1}\beta'^\perp$ is a reflection is DN and ND directions (**Find some justification of this**). The total number of DN and ND directions is denoted as $\#_{ND}$. Now, we prove that $\#_{ND}$ is even. The number of elements of different relevant sets is as follows;

$$n(S_D) = D-p, \;\; n(S_N) = p, \;\; n(S'_D) = D-p', \;\; n(S'_N) = p'$$

Now, we make a Venn diagram of the involved sets and the number of elements in each section of the diagram. The Venn diagram is as follows (where we wrote $\#_{ND}$ simply as $\#$);



where we used the fact that the number of elements in $S_D \cap S'_N$ and $S'_D \cap S_N$ is $\#_{ND}$ and the total number of elements in all sets is D. ξ and μ are undetermined coefficients but they should be non-negative integers because $n(S_N \cap S'_N) = \mu$ and $n(S_N \cap S'_D) = \xi$. Now, we apply the constraints that $n(S_D) = D-p$ and $n(S_N) = p$, Both of these constraints give the same condition i.e.;

$$\mu + \xi = D - p$$

Now, applying the constraints that $n(S'_D) = D-p'$ and $n(S'_N) = p'$ give the same result i.e.;

$$\#_{ND} - \mu + \xi = D - p' \Rightarrow \xi - \mu = D - p' - \#_{ND}$$

So, we get two simultaneous equations for ξ and μ. They are easily solved to give the following;

$$\xi = D - \frac{(p+p') + \#_{ND}}{2}, \;\; \mu = \frac{(p-p') + \#_{ND}}{2}$$

Now, p and p' are both even or odd because they are in type II A theory or type II B theory which have even and odd branes respectively. So, $p+p'$ and $p-p'$ are both even. Therefore, from the expression of μ we deduce that $p-p'+\#_{ND}$ is a non-negative even integer (because μ is a non-negative integer). Since $p-p'$ is even, $\#_{ND}$ is even as well. $\#_{ND}$ is non-negative because it is the number of DN and ND directions. The expression of ξ tells the same story. So, we can write $\#_{ND} = 2j$ where j is a non-negative integer.
Since $\beta^{\perp -1}\beta^{\perp'}$ is the reflection in ND and DN directions, we can pair these directions to give j pairs (since they are even in number) and since a reflection of the coordinate axis of a two-dimensional plane is a rotation by π degrees, we can write $\beta^{\perp -1}\beta^{\perp'}$ as;

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[i\pi\left(J_1 + \ldots + J_j\right)\right]$$

For the spinor representation, J_i's have to be half-integers, and thus $e^{\pi i J}$ has eigenvalues equal to $\pm i$. Now, if j is odd, then we can never have $J_1 + ... + J_j$ equal to an even number (and we need this sum to be an even number to make $\beta^{\perp -1}\beta^{\perp'}$ equal to $e^{2\pi i} = 1$). **Does j being even imply that $\beta^{\perp -1}\beta^{\perp'} = 1$ i.e. is j being even is a necessary condition for $\beta^{\perp -1}\beta^{\perp'} = 1$ but not a sufficient condition?**.
So, $\#_{ND}$ is a multiple of 4 i.e. $\#_{ND} \in \{0, 4, 8\}$. We now count the unbroken supersymmetries corresponding to each case. First take the $\#_{ND} = 8$ case. So, we have;

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[\pi i \left(J_1 + J_2 + J_3 + J_4\right)\right]$$

where J_i's are $\pm 1/2$. Different supersymmetries are counted by counting different $\tilde{Q}_\alpha$'s and different $\tilde{Q}_\alpha$'s are counted by counting different (J_1, J_2, J_3, J_4) tuples such that $J_1 + J_2 + J_3 + J_4$ is even. This sum is even when all J_i's are 1/2 or $-1/2$ which gives us two tuples. This sum is also even when any two J_i's are 1/2 and the other two J_i's are $-1/2$. This second possibility can happen in $\binom{4}{2} = 6$ ways and thus, we have 6 more tuples. So, we have $2 + 6 = 8$ unbroken supersymmetries (**What about the supersymmetries breaking due to first brane?**)
Let's do the $\#_{ND} = 4$ case now. In this case, we have;

$$\beta^{\perp -1}\beta^{\perp'} = \exp\left[\pi i \left(J_1 + J_2\right)\right]$$

So, we need (J_1, J_2) tuples such that $J_1 + J_2$ is even. This can only happen when $(J_1, J_2) = (1/2, -1/2)$ or $(J_1, J_2) = (-1/2, 1/2)$. This gives only two cases. How many spinors does it correspond to? It should correspond to 8 spinors and thus, 8 unbroken supersymmetries. (**How does the factor of 4 appear? Because two other eigenvalues don't matter and we have a factor of 2^2? or some other reason? and again, what about the other brane equation?)**
The $\#_{ND} = 0$ case is trivial. In this case, $\beta^{\perp -1}\beta^{\perp'} = 1$, there is no condition that comes from the second brane. Therefore, the supersymmetries allowed by the first brane are unbroken. This is 16 in number because this case is T-dual to type-I case (**find more justification**).
Do the other proof of $\#_{ND}$ being a multiple of 4 by calculating NS zero point energy.

13.4.1 Branes at general angles

We start with the example of two $D-4$ branes. They are taken to be in $(2, 4, 6, 8)$ directions with separation in 1 direction. To imagine this scenario, the reader can take the $y-$axis to represent $(2, 4, 6, 8)$ collectively and $x-$axis to represent 1 direction. The $z-$axis can be taken to represent $3, 5, 7$ or 9 direction depending on the requirement. Now, we rotate one brane with rotation ϕ_1 in the $(2, 3)$ plane, ϕ_2 in $(4, 5)$ plane, ϕ_3 in $(6, 7)$ plane and ϕ_4 in $(8, 9)$ plane. There is no rotation in any plane involving 1 direction because that will make the separation between the branes position dependent. All of these rotations are collectively called ρ. Then, the unbroken supersymmetry is the one unbroken by the the unrotated brane and the unbroken supersymmetry unbroken by second brane which is (**find justification for this**);

$$Q_\alpha + (\rho^{-1}\beta^\perp \rho Q)_\alpha = Q_\alpha + (\beta^\perp \beta^{\perp -1}\rho^{-1}\beta^\perp \rho Q)_\alpha$$

So, the unbroken supersymmetry is generated by spinors that is left invariant by the following operator;

$$\beta^{\perp -1}\rho^{-1}\beta^\perp \rho = \beta^{\perp -1}\beta^\perp \rho^2 = \rho^2$$

(Find justification of the first step. Is it due to some kind of commutation relation of parity and rotation?). In the spinor representation, this rotation ρ is represented as follows;

$$\rho = \exp\left(i \sum_{a=1}^{4} s_a \phi_a\right) \Rightarrow \rho^2 = \exp\left(2i \sum_{a=1}^{4} s_a \phi_a\right) \quad \text{where } s_a \in \{-1/2, 1/2\} \tag{13.4.3}$$

So, we want this phase to be 1. In other words, we want;

$$2s_1\phi_1 + 2s_2\phi_2 + 2s_3\phi_3 + 2s_4\phi_4 = 0 \pmod{2\pi}$$

where $2s_i$'s can take the values equal to ± 1. We see that if this condition is satisfied for a $(2s_1, ..., 2s_4)$ tuple, then it is also satisfied for $(-2s_1, ..., -2s_4)$ tuple. So, we can focus our attention to the cases where $2s_1 = 1$ so that we don't consider equivalent tuples more than once. We can tabulate all the inequivalent conditions as follows;

s_1	s_2	s_3	s_4	Condition
1	-1	-1	-1	$\phi_1 - \phi_2 - \phi_3 - \phi_4 = 0 \pmod{2\pi}$
1	-1	-1	1	$\phi_1 - \phi_2 - \phi_3 + \phi_4 = 0 \pmod{2\pi}$
1	-1	1	-1	$\phi_1 - \phi_2 + \phi_3 - \phi_4 = 0 \pmod{2\pi}$
1	-1	1	1	$\phi_1 - \phi_2 + \phi_3 + \phi_4 = 0 \pmod{2\pi}$
1	1	-1	-1	$\phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \pmod{2\pi}$
1	1	-1	1	$\phi_1 + \phi_2 - \phi_3 + \phi_4 = 0 \pmod{2\pi}$
1	1	1	-1	$\phi_1 + \phi_2 + \phi_3 - \phi_4 = 0 \pmod{2\pi}$
1	1	1	1	$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0 \pmod{2\pi}$

Some possibilities are considered as follows;

- The last condition allows the $(2s_1, 2s_2, 2s_3, 2s_4) = (1, 1, 1, 1)$ and $(2s_1, 2s_2, 2s_3, 2s_4) = (-1, -1, -1, -1)$ tuples and thus, two supersymmetries are allowed out of the original 16. So, this case breaks seven-eighths of supersymmetry.
- If we want $\phi_1 + \phi_2 + \phi_3 = \phi_4 = 0 \pmod{2\pi}$ then this is satisfied by last two rows in the table and thus, we have $2 \times 2 = 4$ unbroken supersymmetries (a factor of two arises because we also count the equivalent tuples that have $s_1 = -1$).
- If we want $\phi_1 + \phi_2 = \phi_3 + \phi_4 = 0 \pmod{2\pi}$ then this is satisfied by the last row and the fifth row in the table and thus, we have $2 \times 2 = 4$ unbroken supersymmetries.
- If we want $\phi_1 + \phi_2 = \phi_3 = \phi_4 = 0 \pmod{2\pi}$, then this condition is satisfied by the last four rows in the table. Thus, we have $2 \times 4 = 8$ unbroken supersymmetries.
- We now consider the case when k of ϕ_i's are $\pi/2$ and the rest of them are zeros. If $k = 0$ then all the cases in the table are satisfied and thus, $2 \times 8 = 16$ supersymmetries are unbroken. This is the same as $\#_{ND} = 0$ case.
- If $k = 1$ or $k = 3$, then none of the conditions in the table can be satisfied and thus, all the supersymmetries are broken. These can be taken to be the same as $\#_{ND} = 2$ or $\#_{ND} = 6$ cases because this case isn't supersymmetric at all. (**Is it enough to take them as these cases?**).
- If $k = 2$, then there are four rows in the table above which are satisfied. **(Write the details here. You know them. Just write them)**. So, we have $2 \times 4 = 8$ supersymmetries unbroken. This is the same as the case $\#_{ND} = 4$.
- The $k = 4$ case is satisfied for rows $2, 3, 5, 8$. This case also has $2 \times 4 = 8$ unbroken supersymmetries. It can be taken to be the same case $\#_{ND} = 8$ case.

Now, we define the following complex coordinates to make sense of the rotations;

$$Z^1 = X^2 + iX^3, \;\; Z^2 = X^4 + iX^5, \;\; Z^3 = X^6 + iX^7, \;\; Z^4 = X^8 + iX^9$$

We will denote the conjugate of these complex numbers as $\overline{Z^a} = Z^{\bar{a}}$. So, out of the full $SO(8)$ rotation group in eight dimensions, only the $U(4)$ subgroup retains the complex structure. The rotation ρ in particular is;

$$\rho = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4}) \Rightarrow \det \rho = e^{i(\phi_1 + \phi_2 + \phi_3 + \phi_4)}$$

Now, we consider the various cases that we discussed before.

- If $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0 \pmod{2\pi}$ then $\det \rho = 1$ an thus, $\rho \in SU(4)$. This case has 2 unbroken supersymmetries.
- If $\phi_1 + \phi_2 + \phi_3 = \phi_4 = 0 \pmod{2\pi}$ then $\det \rho = 1$ an thus, $\rho \in SU(3)$ ($\phi_4 = 0 \pmod{2\pi}$ and thus, it doesn't give any factor in the group). This case has 8 unbroken supersymmetries.
- If $\phi_1 + \phi_2 = \phi_3 + \phi_4 = 0 \pmod{2\pi}$ then $\det \rho = 1$ an thus, $\rho \in SU(2) \times SU(2)$. This case has 8 unbroken supersymmetries.
- If $\phi_1 + \phi_2 = \phi_3 = \phi_4 = 0 \pmod{2\pi}$ then $\det \rho = 1$ an thus, $\rho \in SU(2)$ ($\phi_3 = \phi_4 = 0 \pmod{2\pi}$ and thus, they don't give any factor in the group). This case has 8 unbroken supersymmetries.

We now write the boundary conditions in terms of the Z^a's. In the unrotated brane version, the brane extends in the following directions;

$$(2,4,6,8) = (\mathrm{Re}(Z^1), \mathrm{Re}(Z^2), \mathrm{Re}(Z^3), \mathrm{Re}(Z^4))$$

and thus the following is true;

$$X^3 = \mathrm{Im}(Z^1) = 0, \;\; X^5 = \mathrm{Im}(Z^2) = 0, \;\; X^7 = \mathrm{Im}(Z^3) = 0, \;\; X^9 = \mathrm{Im}(Z^4) = 0$$

The Neumann boundary conditions for directions parallel to the D-brane are thus, as follows;

$$\partial_1 \mathrm{Re}(Z^1) = 0, \;\; \partial_1 \mathrm{Re}(Z^2) = 0, \;\; \partial_1 \mathrm{Re}(Z^3) = 0, \;\; \partial_1 \mathrm{Re}(Z^4) = 0$$

So, the boundary conditions are as follows;

$$\partial_1 \mathrm{Re}(Z^a) = \mathrm{Im}(Z^a) = 0, \;\; a \in \{1, ..., 4\}$$

In the rotated brane version, Z^a changes to $e^{\phi_a} Z^a$ and thus, if we undo these rotations i.e. if we consider $e^{-\phi_a} Z^a$, the boundary conditions then become the boundary conditions for the unrotated brane. So, the boundary conditions for the rotated brane are as follows;

$$\partial_1 \mathrm{Re}(e^{-\phi_a} Z^a) = \mathrm{Im}(e^{-\phi_a} Z^a) = 0, \;\; a \in \{1, ..., 4\}$$

If we let $\sigma^1 = 0$ endpoint on the unrotated brane and $\sigma^1 = \pi$ endpoint on the rotated brane, then we have the following boundary conditions;

$$\sigma^1 = 0: \;\; \partial_1 \mathrm{Re}(Z^a) = \mathrm{Im}(Z^a) = 0, \;\; a \in \{1, ..., 4\}$$

$$\sigma^1 = \pi: \;\; \partial_1 \mathrm{Re}(e^{-\phi_a} Z^a) = \mathrm{Im}(e^{-\phi_a} Z^a) = 0, \;\; a \in \{1, ..., 4\}$$

These boundary conditions give the following mod expansion (**derive this using the doubling trick**);

$$Z^a(w, \bar{w}) = \mathcal{Z}^a(w) + \overline{\mathcal{Z}^a}(\bar{w}) \text{ where } \mathcal{Z}^a(w) = i\sqrt{\frac{\alpha'}{2}} \sum_{\mathbb{Z}+\nu_a} \frac{\alpha_r^a}{r} e^{irw}$$

where $\nu_a = \phi_a/\pi$. It also gives the following partition function for a single Z scalar (**derive this, the E power of q is a bit unclear**);

$$\mathcal{Z}_{Z,\mathrm{rotated}} = q^{E_0} \prod_{m=0}^{\infty} \left(1 - q^{m+\phi/\pi}\right)^{-1} \left(1 - q^{m+1-\phi/\pi}\right)^{-1} \quad \text{where } E_0 = \frac{1}{24} - \frac{1}{2}\left(\frac{\phi}{\pi} - \frac{1}{2}\right)^2 \tag{13.4.4}$$

Note that if $\phi = 0$, then $E_0 = -1/12$ which one would expect for a Z scalar (because one Z contains two X's and the partition function for one X has the power $q^{-1/24}$). We can write this partition function in terms of ϑ functions as well. For this purpose, consider the definition of $\vartheta_{11}(\nu, \tau)$ with $\tau = it$. We get the following;

$$\vartheta_{11}(\nu, it) = -2e^{-\pi t/4} \sin(\pi\nu) \prod_{m=1}^{\infty} (1 - q^m)(1 - zq^m)(1 - z^{-1}q^m)$$

$$= -2e^{\pi i t/4} q^{-1/24} \sin(\pi\nu) \eta(it) \prod_{m=1}^{\infty} (1 - zq^m)(1 - z^{-1}q^m) \quad \text{where } q = e^{-2\pi t} \;\; z = e^{2\pi i\nu}$$

To match this expression with the expression in (13.4.4), we should choose $z = q^{\phi/\pi}$ and thus, $\nu = i\phi t/\pi$. So, we consider the following;

$$\vartheta_{11}(i\phi t/\pi, it) = -2e^{-\pi t/4} q^{-1/24} \sin(i\phi t) \eta(it) \prod_{m=1}^{\infty} (1 - q^{m+\phi/\pi})(1 - q^{m-\phi/\pi})$$

$$= -2q^{1/8-1/24} \frac{1}{2i} \left(e^{-\phi t} - e^{\phi t}\right) \eta(it) \frac{1}{1 - q^{\phi/\pi}} \prod_{m=0}^{\infty} (1 - q^{m+\phi/\pi})(1 - q^{m+1-\phi/\pi})$$

$$= -iq^{1/8-1/24-\phi/2\pi+E_0}\eta(it)\frac{1}{\mathcal{Z}_{Z\text{ rotated}}}$$

where in the last step, we used (13.4.4). Now, let's calculate the power of q that is appearing;

$$\frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+E_0 = \frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+\frac{1}{24}-\frac{1}{2}\left(\frac{\phi}{\pi}-\frac{1}{2}\right)^2 = \frac{1}{8}-\frac{1}{24}-\frac{\phi}{2\pi}+\frac{1}{24}-\frac{\phi^2}{2\pi^2}-\frac{1}{8}+\frac{\phi}{2\pi} = -\frac{\phi^2}{2\pi^2}$$

and thus, the q factor can be written as follows;

$$q^{1/8-1/24-\phi/2\pi+E_0} = q^{-\phi^2/2\pi^2} = e^{\phi^2 t/\pi}$$

Therefore, we get;

$$\vartheta_{11}(i\phi t/\pi, it) = -ie^{\phi^2 t/\pi}\eta(it)\frac{1}{\mathcal{Z}_{Z\text{ rotated}}} \Rightarrow \mathcal{Z}_{Z\text{ rotated}} = \frac{-ie^{\phi^2 t/\pi}\eta(it)}{\vartheta_{11}(i\phi t/\pi, it)} \tag{13.4.5}$$

Similarly, the fermionic partition function becomes (**derive this and elaborate on the footnote that requires pictures**);

$$Z^\alpha_\beta(\phi, it) = \frac{\vartheta_{\alpha\beta}(i\phi t/\pi, it)}{\exp(\phi^2 t/\pi)\eta(it)} \tag{13.4.6}$$

Since we are talking about the open string only, we have only one fermionic sector and thus, right now, we only need the generalized $Z^+_\psi(it)$ right now. It is as follows;

$$Z^+(\phi, it) = \frac{1}{2}\left(\prod_{a=1}^4 Z^0_0(\phi_a, it) - \prod_{a=1}^4 Z^0_1(\phi_a, it) - \prod_{a=1}^4 Z^1_0(\phi_a, it) - \prod_{a=1}^4 Z^1_1(\phi_a, it)\right)$$

Now, we prove the following generalization of (7.2.2) (**prove it**);

$$\frac{1}{2}\left(\prod_{a=1}^4 Z^0_0(\phi_a, it) - \prod_{a=1}^4 Z^0_1(\phi_a, it) - \prod_{a=1}^4 Z^1_0(\phi_a, it) - \prod_{a=1}^4 Z^1_1(\phi_a, it)\right) = \prod_{a=1}^4 Z^1_1(\phi'_a, it)$$

where;

$$\phi'_1 = \frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4),\ \ \phi'_2 = \frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4)$$
$$\phi'_3 = \frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4),\ \ \phi'_2 = \frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4)$$

Notice using previous analysis that if $\phi'_1 = 0$, then there are only two unbroken supersymmetries but if ϕ'_2,ϕ'_3 or ϕ'_4 is zero, then there might be four or eight unbroken supersymmetries. So, we see that the total partition function coming from all bosonic and fermionic excitations is as follows;

$$\prod_{a=1}^4\frac{-ie^{\phi_a^2 t/\pi}\eta(it)}{\vartheta_{11}(i\phi_a t/\pi, it)}\prod_{a=1}^4 Z^1_1(\phi'_a, it) = \prod_{a=1}^4\frac{-ie^{\phi_a^2 t/\pi}\eta(it)}{\vartheta_{11}(i\phi_a t/\pi, it)}\prod_{a=1}^4\frac{\vartheta_{11}(i\phi\prime_a t/\pi, it)}{\exp(\phi'^2_a t/\pi)\eta(it)}$$
$$= \prod_{a=1}^4\exp\left(\frac{t}{\pi}(\phi_a^2-\phi_a'^2)\right)\prod_{a=1}^4\frac{\vartheta_{11}(i\phi\prime_a t/\pi, it)}{\vartheta_{11}(i\phi_a t/\pi, it)}$$

(What to do with this exponential factor? Also, derive the potential which agrees with this expression if we set $p = 8$ in the parallel brane expression. See Mumford for theta identities). The potential is as follows;

$$V = -\int_0^\infty\frac{dt}{t}(8\pi\alpha' t)^{-1/2}\exp\left(-\frac{ty_1^2}{2\pi\alpha'}\right)\prod_{a=1}^4\frac{\vartheta_{11}(i\phi'_a t/\pi, it)}{\vartheta_{11}(i\phi_a t/\pi, it)} \tag{13.4.7}$$

The dominant contribution comes when y_1 is minimum i.e. when strings are near the point of closest approach of the branes. The sign of the potential is not definite as the following quantity doesn't have a definite sign;

$$-\prod_{a=1}^4\frac{\sin(\phi'_a t)}{\sin(\phi_a t)}$$

sin functions comes the definition of ϑ_{11}. Everything else in the integrand is positive definite. In the definition of $\vartheta_{11}(\nu, it)$, sending $\nu \to -\nu$ keeps the product part invariant but sends $\sin\pi\nu$ to $-\sin\pi\nu$. Thus, $\vartheta_{11}(\nu, it)$ is an odd function of ν and thus, it is zero at $\nu = 0$. Now, we investigate the cases when ϕ_a's or ϕ_a's are zero because the ϑ_{11} functions vanish in these cases.

If $\phi_a = 0$ for some a, then the branes become parallel in some direction. For example, if $\phi_4 = 0$, then the branes become parallel in the 8th direction. The strings can now move in this direction. Since ϕ_4 is the rotation angle in the $8-9$ plane, the 8 and 9 directions become like directions when the branes are parallel with the branes extending in the 8th direction and the 9th direction being perpendicular to the brane. If we look at the partition function of the parallel branes for $p = 4$, the contribution of the $8-9$ directions should be;

$$L\eta(it)^{-2}(8\pi^2\alpha' t)^{-1/2}$$

where L is the length of the noncompact 8th direction However, for $\phi_4 \neq 0$, the contribution of $8-9$ directions to the partition function is in (13.4.5). So, we need to do the following replacement;

$$\frac{-ie^{\phi_4^2 t/\pi}|_{\phi_4=0}\eta(it)}{\vartheta_{11}(i\phi_4 t/\pi, it)} = \frac{-i\eta(it)}{\vartheta_{11}(i\phi_4 t/\pi, it)} \to L\eta(it)^{-2}(8\pi^2\alpha' t)^{-1/2} \Rightarrow \vartheta_{11}(i\phi_a t/\pi, it)^{-1} \to iL\eta(it)^{-3}(8\pi^2\alpha' t)^{-1/2}$$

In order to generalize this picture to branes with some other number of dimensions, we use T duality. If we T dualize in the 8th direction, then the branes loose a dimension and we have 3-branes that are rotated in $2-3$, $4-5$ and $6-7$ directions. These branes might have separations in the 8th and 9th directions. The fermion partition number is unaffected by it (**prove this**). The contribution due to 8th and 9th direction to the partition function will now be (**prove that the $8\pi\alpha' t$ factor won't come**);

$$\eta(it)^{-2}\exp\left[-\frac{t(y_8^2+y_9^2)}{2\pi\alpha'}\right]$$

and thus, we have to do the following replacement;

$$\frac{-ie^{\phi_4^2 t/\pi}|_{\phi_4=0}\eta(it)}{\vartheta_{11}(i\phi_4 t/\pi, it)} = \frac{-i\eta(it)}{\vartheta_{11}(i\phi_4 t/\pi, it)} \to \eta(it)^{-2}\exp\left[-\frac{t(y_8^2+y_9^2)}{2\pi\alpha'}\right]$$

$$\Rightarrow \vartheta_{11}(i\phi_a t/\pi, it)^{-1} \to i\eta(it)^{-3}\exp\left[-\frac{t(y_8^2+y_9^2)}{2\pi\alpha'}\right]$$

T dualizing in the 9th direction will cause the brane to extend in the 9th direction as well. This will obviously give a factor of L_9 i.e length of the 9th direction. It will also give a factor of $(8\pi^2\alpha' t)^{-1/2}$ because we get such a factor for all brane directions (**Why don't we get η? See that**). So, by T dualizing, we can get the potential for branes of different directions (**what about rotated branes of different dimensions? Does 'rotated' mean any sense in this context?**).

The potential in (13.4.7) vanishes if some ϕ_a vanishes. Now, we see what phases in (13.4.3) that are labeled by the tuple $(2s_1, 2s_2, 2s_3, 2s_4)$ are unity when any of the ϕ'_a phases vanish. If $\phi'_1 = 0$, then $(\pm1, \pm1, \pm1, \pm1)$ are unity and two supersymmetries are unbroken. If $\phi'_2 = 0$, then $(\pm1, \pm1, \mp1, \mp1)$ phase is unity and again, two supersymmetries are unbroken. Similarly, for vanishing ϕ'_3 and ϕ'_4, the phases $(\pm1, \mp1, \pm1, \mp1)$ and $(\pm1, \mp1, \mp1, \pm1)$ respectively phases are unity. Again, two supersymmetries are unbroken. However, we see that these phases are only eight in number but in total, there are sixteen $(2s_1, 2s_2, 2s_3, 2s_4)$ phases. There are some phases i.e. when we have an odd number of -1's (4 of them have a single -1 and 4 of them have three -1's) such that if any of these phases are unity, we will have two unbroken supersymmetries (because if $(2s_1, 2s_2, 2s_3, 2s_4)$ is unity then $(-2s_1, -2s_2, -2s_3, -2s_4)$ is also unity) but none of ϕ'_a phases are zero and thus, the potential in (13.4.7) is non-zero. **(Do the expansion large separation part and the tachyon part)**.

13.5 D-brane interactions: dynamics

13.5.1 D-brane scattering

The relative motion of D Branes is described by analytically continuing the angle variable to the rapidity variable. If only ϕ_1 is nonzero so that the rotation is in the $X^2 - X^3$ plane, the following equation is satisfied;

$$X^3 = \tan\phi_1 X^2$$

we define;

$$X^3 \to iX^0, \quad \phi_1 = -iu$$

where u is the rapidity. So, the following is satisfied;

$$X^3 = \tan(-iu)(iX^0) = -i\tanh(u)(iX^0) = \tanh(u)X^0$$

The amplitude of interaction for for general p is given as follows (**derive this**);

$$\mathcal{A} = -iV_p \int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-p/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(ut/2\pi, it)^4}{\eta(it)^9 \vartheta_{11}(ut/\pi, it)} \tag{13.5.1}$$

Using modular transformation, we have;

$$\vartheta_{11}\left(\frac{ut}{2\pi}, it\right) = it^{-1/2} e^{-u^2t^2/4\pi} \vartheta_{11}\left(-\frac{iu}{2\pi}, \frac{i}{t}\right) = -it^{-1/2} e^{-u^2t^2/4\pi} \vartheta_{11}\left(\frac{iu}{2\pi}, \frac{i}{t}\right)$$

Moreover,

$$\vartheta_{11}\left(\frac{ut}{\pi}, it\right) = -it^{-1/2} e^{-u^2t^2/\pi} \vartheta_{11}\left(\frac{iu}{\pi}, \frac{i}{t}\right)$$

and,

$$\eta(it) = t^{-1/2}\eta\left(\frac{i}{t}\right)$$

Using these transformed quantities, we can write the amplitude as follows;

$$\mathcal{A} = -iV_p \int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-p/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) t^{-2} \frac{e^{-u^2t^2/\pi} \vartheta_{11}(iu/2\pi, i/t)^4}{t^{-9/2}\eta(i/t)^9(-i)t^{-1/2}e^{-u^2t^2/\pi}\vartheta_{11}\left(iu/\pi, i/t\right)}$$

$$= \frac{V_p}{(8\pi^2\alpha')^{p/2}} \int_0^\infty \frac{dt}{t} t^{(6-p)/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)}$$

We can write this amplitude as an integral over a worldline. For that purpose, we manipulate this amplitude as follows;

$$\mathcal{A} = \frac{V_p}{(8\pi^2\alpha')^{p/2}} \int_0^\infty \frac{dt}{t} t^{(6-p)/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)}$$

$$= \frac{2\pi\sqrt{2\alpha'}V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{1/2}} t^{(5-p)/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)}$$

$$= \frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \left(\exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\pi}{\tanh u}\sqrt{\frac{2\alpha'}{t}}\right)$$

Now, the factor in the last parenthesis can be written as follows;

$$\exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\pi}{\tanh u}\sqrt{\frac{2\alpha'}{t}} = \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \int_{-\infty}^\infty d\tau \exp\left(-\frac{t\tau^2\tanh^2 u}{2\pi\alpha'}\right) = \int_{-\infty}^\infty d\tau \exp\left(-\frac{t(y^2 + \tau^2\tanh^2 u)}{2\pi\alpha'}\right)$$

$$= \int_{-\infty}^\infty d\tau \exp\left(-\frac{t(y^2+\tau^2 v^2)}{2\pi\alpha'}\right) = \int_{-\infty}^\infty d\tau \exp\left(-\frac{tr^2}{2\pi\alpha'}\right)$$

where

$$v = \tanh u, \quad r^2(\tau) = y^2 + v^2\tau^2$$

So, we get;

$$\mathcal{A} = \frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \int_{-\infty}^\infty d\tau \exp\left(-\frac{tr^2}{2\pi\alpha'}\right)$$

$$= -i\int_{-\infty}^\infty d\tau \frac{2iV_p}{(8\pi^2\alpha')^{(p+1)/2}} \int_0^\infty dt\, t^{(5-p)/2} \frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9 \vartheta_{11}(iu/\pi, i/t)} \exp\left(-\frac{tr^2}{2\pi\alpha'}\right)$$

$$= -i\int_{-\infty}^{\infty} d\tau V(r,\tau)$$

where

$$V(r,\tau) = i\frac{2V_p}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(5-p)/2}\frac{\tanh u\, \vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9\vartheta_{11}(iu/\pi, i/t)}\exp\left(-\frac{tr^2}{2\pi\alpha'}\right) \tag{13.5.2}$$

Now, the expansion of $\vartheta_{11}(\nu,\tau)$ is done as follows;

$$\vartheta_{11}(\nu,\tau) = -2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m)$$

$$= -2e^{\pi i\tau/4}\left(\pi\nu + \frac{\pi^3\nu^3}{6} + ...\right)\prod_{m=1}^{\infty}(1-q^m)(1-q^m-2\pi i\nu q^m+...)(1-q^m+2\pi i\nu q^m+...)$$

$$= -2e^{\pi i\tau/4}\nu\pi\left(1+\frac{\pi^2\nu^2}{6}+\mathcal{O}(\nu^2)\right)\prod_{m=1}^{\infty}(1-q^m)\prod_{m=1}^{\infty}\left[(1-q^m)^2+(1-q^m)2\pi i\nu q^m-(1-q^m)2\pi i\nu q^m+\mathcal{O}(\nu^2)\right]$$

$$= -2\nu\pi e^{\pi i\tau/4}q^{-1/8}\eta(\tau)^3+\mathcal{O}(\nu^2) = -2\pi\nu\eta(\tau)^3+\mathcal{O}(\nu^2)$$

This result implies;

$$\vartheta_{11}\left(\frac{iu}{2\pi},\frac{i}{t}\right) = -2\pi\eta(i/t)^3\frac{iu}{2\pi}+\mathcal{O}(u^2) = -iu\,\eta(i/t)^3+\mathcal{O}(u^2)$$

Similarly, we have;

$$\vartheta_{11}\left(\frac{iu}{\pi},\frac{i}{t}\right) = -2\pi\eta(i/t)^3\frac{iu}{\pi}+\mathcal{O}(u^2) = -2iu\,\eta(i/t)^3+\mathcal{O}(u^2)$$

Since $v = \tanh u = u + \mathcal{O}(u^3)$, we have $v \sim u$ for small u. So, we have;

$$\vartheta_{11}\left(\frac{iu}{2\pi},\frac{i}{t}\right) = -iv\,\eta(i/t)^3+\mathcal{O}(v^2)$$

Similarly, we have;

$$\vartheta_{11}\left(\frac{iu}{\pi},\frac{i}{t}\right) = -2iv\,\eta(i/t)^3+\mathcal{O}(v^2)$$

Using these expansions, (13.5.2) becomes;

$$V(r,v) = -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(5-p)/2}e^{-tr^2/2\pi\alpha'}+\mathcal{O}(v^6)$$

$$= -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{r^2}\right)^{(7-p)/2}\int_0^\infty d\mu\,\mu^{(5-p)/2}e^{-\mu}+\mathcal{O}(v^6)$$

$$= -v^4\frac{V_p}{(8\pi^2\alpha')^{(p+1)/2}}\left(\frac{2\pi\alpha'}{r^2}\right)^{\frac{7-p}{2}}\Gamma\left(\frac{7-p}{2}\right)+\mathcal{O}(v^6) = -\frac{v^4}{r^{7-p}}\frac{V_p}{\alpha'^{p-3}}2^{2-2p}\pi^{(5-3p)/2}\Gamma\left(\frac{7-p}{2}\right)+\mathcal{O}(v^6) \tag{13.5.3}$$

Using these results, we see that the leading v term in (13.5.2) is proportional to v^4. **(Do the argument about v^2 term)**. Now, we investigate the small r limit. Since we have $-tr^2/2\pi\alpha'$ in the exponential in (13.5.2), the dominant contribution comes when $t \sim 2\pi\alpha'/r^2$. So, small r corresponds to large t i.e. from the light open states (**see more about this**). To see the small r behavior, we expand (13.5.2) in large t i.e. in powers of $1/t$ but don't expand in v (I think it is for small v) (**derive this**). We get;

$$V(\nu,\tau) \sim -2V_p\int_0^\infty \frac{dt}{(8\pi\alpha' t)^{(p+1)/2}}\exp\left(-\frac{tr^2}{2\pi\alpha'}\right)\frac{\tanh u\sin^4 ut/2}{\sin ut}$$

As we saw in (13.5.3), the structure of the v^4 term is v^4/r^{7-p}. The higher terms will have higher powers of v and higher powers of r in the denominator (**check this**). So, no matter how small v is, at small enough r, the dominant term will not be v^4 term. The integrand is smoothed at $ut \approx 1$ (**find justification for this**). Since $u \approx v$ for small v, and since the dominant contribution comes when $t \approx 2\pi\alpha'/r^2$, we have;

$$ut \sim 1 \Rightarrow \frac{2\pi\alpha' v}{r^2} \sim \frac{\alpha' v}{r^2} \sim 1 \Rightarrow r \sim \sqrt{\alpha' v} = l_s\sqrt{v} \tag{13.5.4}$$

So, the slow-moving D brane is relevant for probing distances small than the string length l_s. The time that it takes to scatter δt is;

$$\delta t \sim \frac{r}{v} \sim \sqrt{\frac{\alpha'}{v}}$$

We will denote the uncertainty in distance as δx. For a D brane with velocity v, it should be of the order $\sqrt{\alpha' v}$ as found previously. So, we have the following;

$$\delta x \delta t \sim \sqrt{\frac{\alpha'}{v}} \sqrt{\alpha' v} = \alpha'$$

In general, the uncertainty can't be smaller than δx and thus, the value of $\delta x \delta t$ calculated is a lower bound. Thus, we have **(Look into it more)**;

$$\delta x \delta t \geq \alpha' \tag{13.5.5}$$

This uncertainty relation between coordinates hints at non-commutative geometry.
We can find a minimum distance that a $D0$ brane can probe. Since it is a particle, the wavepacket that minimizes the uncertainty in the Heisenberg uncertainty relation is the Gaussian wavepacket. For the gaussian wavepacket, we have the following;

$$\delta x \delta p \sim \hbar = 1 \Rightarrow \delta x \delta p \sim 1$$

But, for small v, we have $\delta p = m \delta v$ and thus, we have;

$$m \delta x \delta v \sim 1 \Rightarrow \delta x \sim \frac{1}{m \delta v}$$

Now, we use (13.3.11) with $p = 0$ to get the mass of the D0 brane. We get;

$$m = \tau_0 = \alpha'^{-1/2} g^{-1} \Rightarrow \delta x \sim \frac{g\sqrt{\alpha'}}{\delta v}$$

So, the uncertainty relation for the $D0$ brane should be;

$$\delta x \ \delta v \geq g\sqrt{\alpha'} \tag{13.5.6}$$

Since v is small, we can take $\delta v \sim v$ (**Is this assumption right?**) and thus, we have;

$$\delta x \geq \frac{g\sqrt{\alpha'}}{v}$$

Moreover, we have an inequality from (13.5.4) which reads;

$$\delta x \geq \sqrt{\alpha' v}$$

Imposing these two inequalities gives us a region and the lowest possible value of δx in this region is found when we have;

$$\frac{g\sqrt{\alpha'}}{v} = \sqrt{\alpha' v} \Rightarrow v^{3/2} = g \Rightarrow v = g^{2/3}$$

Using this value of v in the inequality corresponding to (13.5.4), we get;

$$\delta x \geq \alpha'^{1/2} g^{1/3} \tag{13.5.7}$$

13.5.2 D0 brane quantum mechanics

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13.5.3 $\#_{ND} = 4$ system

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13.6 D-brane interactions: bound states

13.6.1 FD bound states

We start with p F strings and q D strings. We assume that all of them are extended in the 1 direction and are in rest. In this case, we have $P_M = -M\delta_{M,0}$, $Q^{NS}_M = pL_1\delta_{M,1}$ and $Q^R_M = qL_1\delta_{M,1}$ where L_1 is the length of the system. The susy algebra in (13.2.1) to (13.2.2) becomes;

$$\{Q_\alpha, Q^\dagger_\beta\} = -2M(\Gamma^0)^2_{\alpha\beta} + \frac{2pL_1}{2\pi\alpha'}(\Gamma^0\Gamma^1)_{\alpha\beta} = 2M\delta_{\alpha\beta} + \frac{2pL_1}{2\pi\alpha'}(\Gamma^0\Gamma^1)_{\alpha\beta}$$

$$\{\tilde{Q}_\alpha, \tilde{Q}^\dagger_\beta\} = -2M(\Gamma^0)^2_{\alpha\beta} - \frac{2pL_1}{2\pi\alpha'}(\Gamma^0\Gamma^1)_{\alpha\beta} = 2M\delta_{\alpha\beta} - \frac{2pL_1}{2\pi\alpha'}(\Gamma^0\Gamma^1)_{\alpha\beta}$$

$$\{Q_\alpha, \tilde{Q}^\dagger_\beta\} = 2\frac{\tau_1}{1!}L_1 q(\beta^1\Gamma^0)_{\alpha\beta} = 2\frac{1}{2\pi\alpha' g}L_1 q(\beta^1\Gamma^0)_{\alpha\beta} = 2\frac{L_1 q}{2\pi\alpha' g}(\Gamma^0\Gamma^1)_{\alpha\beta}$$

(**justify the last step by calculating** $\beta^1\Gamma^0$). These results can be summarized as follows;

$$\frac{1}{2}\left\{\begin{pmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{pmatrix}, \begin{pmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{pmatrix}\right\} = M\delta_{\alpha\beta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{L_1}{2\pi\alpha'}(\Gamma^0\Gamma^1)_{\alpha\beta}\begin{pmatrix} p & q/g \\ q/g & -p \end{pmatrix} \tag{13.6.1}$$

The left side of the equation has non-negative eigenvalues (**write more about it**) and thus, the eigenvalues of the right side must also be non-negative. We now notice that;

$$S_0 = \Gamma^{0+}\Gamma^{0-} - \frac{1}{2} = \frac{1}{4}(\Gamma^0 + \Gamma^1)(-\Gamma^0 + \Gamma^1) - \frac{1}{2} = \frac{1}{4}(2 + \Gamma^0\Gamma^1 - \Gamma^1\Gamma^0) - \frac{1}{2} = \frac{1}{2}\Gamma^0\Gamma^1$$

Since we know that the eigenvalues of S_0 are $\pm 1/2$ and thus, the eigenvalues of $\Gamma^0\Gamma^1$ are ± 1. Therefore, the eigenvalues of the right-hand side of (13.6.1) are as follows;

$$M \pm \frac{L_1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g^2}}$$

(Address the issue of eigenvalues of block diagonal matrix). Since these eigenvalues should be non-negative, we get a bound (i.e. the BPS bound) on the mass from the negative sign case. We have;

$$\frac{M}{L_1} \geq \frac{1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g^2}} \tag{13.6.2}$$

The mass per unit length for p F strings and q D strings can be denoted as $M(p,q)$ and it is given as follows;

$$M(p,q) = \frac{p}{2\pi\alpha'} + \frac{q}{2\pi\alpha' g}$$

We have the following cases;

$$M(1,0) = \frac{1}{2\pi\alpha'} \geq \frac{1}{2\pi\alpha'}$$

$$M(0,1) = \frac{1}{2\pi\alpha' g} \geq \frac{1}{2\pi\alpha' g}$$

$$M(1,1) = \frac{1}{2\pi\alpha'} + \frac{1}{2\pi\alpha' g} = \frac{1 + g^{-1}}{2\pi\alpha'} \geq \frac{\sqrt{1 + g^{-2}}}{2\pi\alpha'}$$

$M(1,1)$ case exceeds the BPS bound and the BPS bound is saturated only if $g \to \infty$.
(Complete the section)

13.6.2 $D0-Dp$ BPS bound

We again derive the BPS bound for this system. Let Dp brane be extending in $(1,...,p)$ directions. In the $D0$ and Dp system, $Q^{NS}=0$ and there are two Ramond charges. We thus have the following susy algebra;

$$\{Q_\alpha, Q^\dagger_\beta\} = -2M(\Gamma^0)^2_{\alpha\beta} = 2M\delta_{\alpha\beta}$$

$$\{\tilde{Q}_\alpha, \tilde{Q}^\dagger_\beta\} = -2M(\Gamma^0)^2_{\alpha\beta} = 2M\delta_{\alpha\beta}$$

$$\{Q_\alpha, \tilde{Q}^\dagger_\beta\} = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \frac{\tau_p}{p!}Q^R_{M_1...M_p}(\beta^{M_1}...\beta^{M_p}\Gamma^0)_{\alpha\beta}\right) = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \tau_p Q^R_{1...p}(\beta\Gamma^0)_{\alpha\beta}\right)$$

where in the last step, we summed over $M_1...M_p$ indices (**include some explanation of this**). Moreover, $\beta = \beta^1...\beta^p$. We can wrap the Dp brane on p torus with volume V_p. So, we have $Q^R_{1,...,p} = V_p$ (**confusion about the charge getting smaller**). Therefore, we get;

$$\{Q_\alpha, \tilde{Q}^\dagger_\beta\} = 2\left(\tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\beta\Gamma^0)_{\alpha\beta}\right) = 2Z_{\alpha\gamma}\Gamma^0_{\gamma\beta}$$

where $Z_{\alpha\gamma} = \tau_0\delta_{\alpha\gamma} + \tau_p V_p\beta_{\alpha\gamma}$. We also get the following (**include its derivation**);

$$\{\tilde{Q}_\alpha, Q^\dagger_\beta\} = -2Z^\dagger_{\alpha\gamma}\Gamma^0_{\gamma\beta}$$

The susy algebra can be written as follows;

$$\frac{1}{2}\left\{\begin{pmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{pmatrix}, \begin{pmatrix} Q^\dagger_\beta & \tilde{Q}^\dagger_\beta \end{pmatrix}\right\} = M\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha\beta} + \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0 \end{pmatrix}\Gamma^0_{\gamma\beta} \tag{13.6.3}$$

The eigenvalues of the right-hand side should again be non-negative and thus, we get;

$$M\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha\beta} + \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0 \end{pmatrix}\Gamma^0_{\gamma\beta} \geq 0 \Rightarrow M\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha\beta} \geq -\begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0 \end{pmatrix}\Gamma^0_{\gamma\beta}$$

$$\begin{aligned} \Rightarrow M^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha\beta} &\geq \begin{pmatrix} 0 & Z_{\alpha\gamma} \\ -Z^\dagger_{\alpha\gamma} & 0 \end{pmatrix}\Gamma^0_{\gamma\sigma}\begin{pmatrix} 0 & Z_{\sigma\lambda} \\ -Z^\dagger_{\sigma\lambda} & 0 \end{pmatrix}\Gamma^0_{\lambda\beta} \\ &= \begin{pmatrix} -Z_{\alpha\gamma}\Gamma^0_{\gamma\sigma}Z^\dagger_{\sigma\lambda}\Gamma^0_{\lambda\beta} & 0 \\ 0 & -Z^\dagger_{\alpha\gamma}\Gamma^0_{\gamma\sigma}Z_{\sigma\lambda}\Gamma^0_{\lambda\beta} \end{pmatrix} \end{aligned} \tag{13.6.4}$$

Now, we do the following manipulation;

$$\begin{aligned} (Z^\dagger\Gamma^0)_{\alpha\beta} &= \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\beta^\dagger\Gamma^0)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\beta^{\dagger p}...\beta^{\dagger 1}\Gamma^0)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\beta^p...\beta^1\Gamma^0)_{\alpha\beta} \\ &= \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\Gamma^0\beta^p...\beta^1)_{\alpha\beta} = \tau_0\Gamma^0_{\alpha\beta} + \tau_p V_p(\Gamma^0\beta^{\dagger p}...\beta^{\dagger 1})_{\alpha\beta} = (\Gamma^0 Z^\dagger)_{\alpha\beta} \end{aligned} \tag{13.6.5}$$

where I used the following results in the manipulation above (**is the first identity true for all p?**);

$$\beta^{\dagger m} = (\Gamma\Gamma^m)^\dagger = \Gamma^{m\dagger}\Gamma^\dagger = \Gamma^0\Gamma^m\Gamma^0\Gamma^0 = \Gamma\Gamma^m = \beta^m \tag{13.6.6}$$

$$\beta^m\Gamma^0 = \Gamma\Gamma^m\Gamma^0 = \Gamma^0\Gamma\Gamma^m = \Gamma^0\beta^m \tag{13.6.7}$$

Similarly, we can also derive the following;

$$(Z\Gamma^0)_{\alpha\beta} = (\Gamma^0 Z)_{\alpha\beta} \tag{13.6.8}$$

Using (13.6.5) and (13.6.8) in (13.6.4), we get;

$$M^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{\alpha\beta} \geq \begin{pmatrix} (ZZ^\dagger)_{\alpha\beta} & 0 \\ 0 & (Z^\dagger Z)_{\alpha\beta} \end{pmatrix}$$

where we can easily see that;

$$Z = \tau_0 + \tau_p V_p\beta, \quad Z^\dagger = \tau_0 + \tau_p V_p\beta^\dagger \Rightarrow ZZ^\dagger = \tau_0^2 + \tau_0\tau_p V_p(\beta + \beta^\dagger) + \tau_p^2 V_p^2\beta\beta^\dagger \tag{13.6.9}$$

Now, we notice the following result;

$$\beta^\dagger = \left(\beta^1...\beta^p\right)^\dagger = \beta^{\dagger p}...\beta^{\dagger 1} = \beta^p...\beta^1 = (-1)^{p(p-1)/2}\beta^1...\beta^p \tag{13.6.10}$$

where in the last step, I used the fact that;

$$\beta^m\beta^n = \Gamma\Gamma^m\Gamma\Gamma^n = -\Gamma\Gamma^m\Gamma\Gamma^n = -\beta^n\beta^m \text{ for } m \neq n$$

and I also used the following fact;

$$1 + 2 + ... + (p-1) = \frac{p(p-1)}{2}$$

From (13.6.10), we can see that β is hermitian is p is a multiple of 4. Moreover, we also note that;

$$(\beta^m)^2 = \Gamma\Gamma^m\Gamma\Gamma^m = -\Gamma\Gamma^m\Gamma^m\Gamma = -1 \Rightarrow \beta^2 = \beta^1...\beta^p\beta^1...\beta^p = (-1)^{p(p-1)/2}\beta^p...\beta^1\beta^1...\beta^p$$
$$= (-1)^{p(p-1)/2}(-1)^p = (-1)^{p(p+1)/2}$$

We see that if p is a multiple of 4, then $\beta^2 = 1$. So, we see that for p a multiple of 4, (13.6.9) becomes;

$$ZZ^\dagger = Z^\dagger Z = \tau_0^2 + 2\tau_0\tau_pV_p\beta + \tau_p^2V_P^2 = \left(\tau_0 + \tau_pV_p\beta\right)^2$$

This implies that the BPS bound is as follows (**the beta factor problem**);

$$M \geq \tau_0 + \tau_pV_p \tag{13.6.11}$$

We note that a $D0$ and Dp state saturates this BPS bound and thus, some supersymmetry is unbroken. To make sense of this result, we note that for a $D0-Dp$ system, the only Dirichlet directions can be the p directions on the Dp brane and all of these directions have to be Neumann directions on the $D0$ brane. So, in this case, $\#_{ND} = p$. We saw that if $\#_{ND}$ is a multiple of 4, then some supersymmetry is unbroken. This is exactly what we are seeing here i.e. for p a multiple of 4, the BPS bound is saturated.
Similarly, we can easily see that if p is even but not a multiple of 4 (i.e. if $p = 4k+2$), then we have;

$$\beta^\dagger = -\beta, \quad \beta^2 = -1$$

In this case, (13.6.9) becomes;

$$ZZ^\dagger = Z^\dagger Z = \tau_0^2 + \tau_p^2V_p^2$$

and thus, the BPS bound is easily seen to be;

$$M^2 \geq \tau_0^2 + \tau_p^2V_p^2 \Rightarrow M \geq \sqrt{\tau_0^2 + \tau_p^2V_p^2}$$

This bound is not saturated by a $D0-Dp$ system and it is consistent with the fact that for $\#_{ND} = 4k+2$, the system is not supersymmetric.

13.6.3 $D0-D0$ bound states

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13.6.4 $D0-D2$ bound states

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13.6.5 $D0-D4$ bound states

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13.6.6 D-branes as instantons

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13.6.7 $D0-D6$ bound states

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13.6.8 $D0-D8$ bound states

14 Chapter 14: Strings at strong coupling

14.1 Type IIB string and $SL(2,\mathbb{Z})$ duality

To see the $SL(2,\mathbb{Z})$ duality of type IIB theory, we consider a D string. Its worldsheet is two-dimensional. Like the case for D branes, the excitations longitudinal to the D brane world volume correspond to the gauge field. However, in two dimensions, the gauge field has no dynamics (**expand on this**). So, the only excitations are transverse excitations. The Dirac equation for these excitations is (**make sense of this**);

$$\left(\Gamma^0\partial_0 + \Gamma^1\partial_1\right)u = 0 \tag{14.1.1}$$

Multiplying from the left by Γ^1, we have;

$$\left(\Gamma^1\Gamma^0\partial_0 + (\Gamma^1)^2\partial_1\right)u = 0 \Rightarrow \Gamma^0\Gamma^1\partial_0 u = \partial_1 u \tag{14.1.2}$$

For left and right-handed spinors, we have the following;

$$\left(\partial_0 \mp \partial_1\right)u = 0 \Rightarrow \partial_1 u = \pm\partial_0 u \tag{14.1.3}$$

Using (14.1.3) in (14.1.2), we have;

$$\Gamma^0\Gamma^1\partial_0 u = \pm\partial_0 u \Rightarrow \Gamma^0\Gamma^1 u = \pm u$$

(**Think of the constant of integration here.**) From appendix B, we know that $2S_0 = \Gamma^0\Gamma^1$. Therefore, we see that the s_0 eigenvalue of u has to be $\pm 1/2$. Therefore, the Majorana Weyl fermion living in **16** decomposes as follows;

$$16 \to \left(\frac{1}{2}, \mathbf{8}\right) \oplus \left(-\frac{1}{2}, \mathbf{8}'\right)$$

Infinite string has the same decomposition (**complete**).

14.1.1 $SL(2,\mathbb{Z})$ duality

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14.1.2 The IIB NS5 brane

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14.1.3 $D3$ branes and Montonen Olive duality

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14.2 U-duality

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14.2.1 U-duality and bound states

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14.3 $SO(32)$ type I-heterotic duality

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14.3.1 Quantitative tests

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14.3.2 Type I $D5$ branes

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14.4 Type IIA string and M theory

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14.4.1 U duality and F theory

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14.4.2 IIA branes from 11 dimensions

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14.5 $E_8 \times E_8$ heterotic string

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14.6 What is string theory

14.7 Is M for matrix?

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14.7.1 The M-theory membrane

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14.7.2 Finite n and compactification

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14.8 Black Hole quantum mechanics

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14.8.1 A correspondence principle

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14.8.2 The information paradox

15 Chapter 15: Advanced CFT

15.1 Representations of Virasoro algebra

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15.2 The conformal bootstrap

tt

15.3 Minimal models

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15.3.1 Feigin-Fuchs representation

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15.4 Current algebras

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15.4.1 Modular invariance

tt

15.4.2 Strings on group manifolds

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15.5 Coset models

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15.6 Representations of the $N = 1$ superconformal algebra

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15.7 Rational CFT

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15.8 Renormalization group flows

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15.8.1 Scale invariance and renormalization group flows

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15.8.2 The Zamolodchikov c theorem

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15.8.3 Conformal perturbation theory

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15.9 Statistical Mechanics

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15.9.1 Landau-Ginzburg models

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16 Chapter 16: Orbifolds

16.1 Orbifolds of the heterotic string

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16.1.1 Modular invariance

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16.1.2 Other free CFTs

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16.2 Spacetime supersymmetry

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16.3 Examples

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16.3.1 Connection with grand unification

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16.3.2 Generalizations

tt

16.3.3 World sheet supersymmetries

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16.4 Low energy field theory

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16.4.1 Untwisted states

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16.4.2 T duality

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16.4.3 Twisted states

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16.4.4 Threshold corrections

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17 Chapter 17: Calabi-Yau compactification

17.1 Conditions of $N = 1$ supersymmetry

17.2 Calabi-Yau manifolds

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17.2.1 Real manifolds

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17.2.2 Complex manifolds

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17.2.3 Kahler manifolds

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17.2.4 Manifolds of $SU(3)$ holonomy

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17.2.5 Examples

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17.2.6 Worldsheet supersymmetry

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17.3 Massless spectrum

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17.4 Low energy field theory

tt

17.5 Higher corrections

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17.5.1 Instanton corrections

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17.6 Generalizations

18 Chapter 18: Physics in four dimensions

18.1 Continuous and discrete symmetries

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18.1.1 P,C,T and all that

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18.1.2 The strong CP problem

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18.2 Gauge symmetries

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18.2.1 Gauge and gravitational couplings

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18.2.2 Gauge quantum numbers

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18.2.3 Right moving gauge symmetries

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18.2.4 Gauge symmetries of type II strings

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18.3 Mass scales

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18.4 More on unification

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18.4.1 Conditions for spacetime supersymmetry

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18.5 Low energy actions

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18.6 Supersymmetry breaking in perturbation theory

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18.6.1 Supersymmetry breaking at tree level

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18.6.2 Supersymmetry breaking in the loop expansion

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18.7 Supersymmetry beyond perturbation theory

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18.7.1 An example

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18.7.2 Another example

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18.7.3 Discussion

19 Chapter 19: Advanced Topics

19.1 The $N = 2$ superconformal algebra

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19.1.1 Heterotic string vertex operators

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19.1.2 Chiral primary fields

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19.1.3 Spectral flow

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19.2 Type II strings on Calabi-Yau manifolds

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19.2.1 Low energy actions

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19.2.2 Chiral rings

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19.2.3 Topological string theory

tt

19.3 Heterotic string theories with $(2, 2)$ SCFT

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19.3.1 More on the low energy action

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19.4 $N = 2$ minimal models

tt

19.4.1 Landau Ginzburg models

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19.5 Gepner models

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19.5.1 Connection to Calabi-Yau compactification

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19.6 Mirror symmetry and applications

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19.6.1 Moduli spaces

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19.6.2 The flop

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19.7 The conifold

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19.7.1 The conifold transition

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19.8 String theories on $K3$

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19.9 String duality below 10 dimensions

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19.9.1 Heterotic strings in $7 \leq d \leq 9$

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19.9.2 Heterotic-type II A duality in six dimensions

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19.9.3 Heterotic S-duality in four dimensions

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19.10 Conclusion

It was fun.

20 Appendix A: A short course on path integrals

20.1 Bosonic fields

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20.1.1 Relation to Hilbert space formalism

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20.1.2 Euclidean path integrals

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20.1.3 Diagrams and determinants

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20.1.4 An example

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20.2 Fermionic fields

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21 Appendix B: Spinors and supersymmetry in various dimensions

21.1 Spinors in various dimensions

Suppose that we are working in even dimensions i.e. $d = 2k + 2$ (where $k \in \{0, 1, 2, ...\}$. Let Γ^μ be the gamma matrices in d dimensions. Then, the Clifford algebra is satisfied;

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{21.1.1}$$

Now, we can make creation and annihilation operators from these matrices as follows;

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1)$$

$$\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}) \quad \text{where } a = 1, ..., k \tag{21.1.2}$$

where $\Gamma^{a\pm}$ satisfy the following anti commutation relations;

$$\{\Gamma^{a+}, \Gamma^{b+}\} = \frac{1}{4}\left\{\Gamma^{2a} + i\Gamma^{2a+1}, \Gamma^{2b} + i\Gamma^{2b+1}\right\} = 0$$

$$\{\Gamma^{a-}, \Gamma^{b-}\} = \frac{1}{4}\left\{\Gamma^{2a} - i\Gamma^{2a+1}, \Gamma^{2b} - i\Gamma^{2b+1}\right\} = 0$$

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \frac{1}{4}\left\{\Gamma^{2a} + i\Gamma^{2a+1}, \Gamma^{2b} - i\Gamma^{2b+1}\right\} = \delta^{ab} \tag{21.1.3}$$

The anti-commutation relations involving $\Gamma^{0\pm}$ are as follows;

$$\{\Gamma^{0\pm}, \Gamma^{a+}\} = \{\Gamma^{0\pm}, \Gamma^{a-}\} = \{\Gamma^{0+}, \Gamma^{0+}\} = \{\Gamma^{0-}, \Gamma^{0-}\} = 0$$

$$\{\Gamma^{0+}, \Gamma^{0-}\} = \frac{1}{4}\{\Gamma^0 + \Gamma^1, -\Gamma^0 + \Gamma^1\} = 1 + \frac{1}{2}\Gamma^0\Gamma^1 \tag{21.1.4}$$

(21.1.3) implies that $(\Gamma^{a+})^2 = (\Gamma^{a-})^2 = (\Gamma^{0+})^2 = (\Gamma^{0-})^2 = 0$. Now, let ξ be a spinor that doesn't contain any factor of Γ^{a-} in its expression, then we can make a spinor ζ as follows;

$$\zeta = \Gamma^{k-}\Gamma^{(k-1)-}...\Gamma^{0-}\xi \tag{21.1.5}$$

So, ζ is a spinor annihilated by all Γ^{a-}. We can now make a spinor basis by applying Γ^{a+} on ζ (one Γ^{a+} is applied at most once). A member of this spinor basis is written as follows;

$$\zeta^{\mathrm{s}} = (\Gamma^{k+})^{s_k+1/2}....(\Gamma^{0+})^{s_0+1/2}\zeta \tag{21.1.6}$$

where $s_a = -1/2$ means that the corresponding Γ^{a+} has been applied and $s_a = 1/2$ means that the corresponding Γ^{a+} is applied. Moreover, s is the following k-tuple;

$$\mathrm{s} = (s_0, ..., s_k)$$

Lorentz generators in the spinor representation are given as follows;

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] \tag{21.1.7}$$

which satisfy the Lorentz algebra;

$$i[\Sigma^{\mu\nu}, \Sigma^{\alpha\beta}] = \eta^{\mu\alpha}\Sigma^{\nu\beta} - \eta^{\mu\beta}\Sigma^{\nu\alpha} - \eta^{\nu\alpha}\Sigma^{\mu\beta} + \eta^{\nu\beta}\Sigma^{\mu\alpha} \tag{21.1.8}$$

We can take out a set of mutually commuting Σ's as follows;

$$[\Sigma^{2a,2a+1}, \Sigma^{2b,2b+1}] = i\eta^{2a,2b}\Sigma^{2a+1,2b+1} = i\delta^{2a,2b}\Sigma^{2a+1,2b+1} = 0$$

These Σ's are calculated as follows;

$$\Sigma^{0,1} = -\frac{i}{4}[\Gamma^0, \Gamma^1] = -\frac{i}{2}\Gamma^0\Gamma^1 = -i\left(\Gamma^{0+}\Gamma^{0-} - \frac{1}{2}\right)$$

$$\Sigma^{2a,2a+1} = -\frac{i}{4}[\Gamma^{2a}, \Gamma^{2a+1}] = -\frac{i}{2}\Gamma^{2a}\Gamma^{2a+1} = \Gamma^{(2a)+}\Gamma^{(2a+1)-} - \frac{1}{2}$$

So, we can define the following convenient operator;

$$S_a = \begin{cases} \Sigma^{2a,2a+1} \text{ for } a = 1, ..., k \\ i\Sigma^{0,1} \text{ for } a = 0 \end{cases} = i^{a,0}\Sigma^{2a,2a+1} = \Gamma^{(2a)+}\Gamma^{(2a+1)-} - \frac{1}{2} \tag{21.1.9}$$

Now, we calculate $S_a\zeta^{\mathsf{s}}$ as follows;

$$S_a\zeta^{\mathsf{s}} = \left(\Gamma^{a+}\Gamma^{a-} - \frac{1}{2}\right)(\Gamma^{k+})^{s_k+1/2}...(\Gamma^{a+})^{s_a+1/2}...(\Gamma^{0+})^{s_0+1/2}\zeta^{\mathsf{s}}$$

$$= (\Gamma^{k+})^{s_k+1/2}...\Gamma^{a+}\Gamma^{a-}(\Gamma^{a+})^{s_a+1/2}...(\Gamma^{0+})^{s_0+1/2}\zeta^{\mathsf{s}} - \frac{1}{2}\zeta^{\mathsf{s}} \tag{21.1.10}$$

Now, we need to evaluate $\Gamma^{a-}(\Gamma^{a+})^{s_a+1/2}$ for different values of s_a. Here it is;

$$\begin{cases} \Gamma^{a-}(\Gamma^{a+})^{s_a+1/2} = \Gamma^{a-} \quad \text{if } s_a = -\frac{1}{2} \\ \Gamma^{a-}(\Gamma^{a+})^{s_a+1/2} = \Gamma^{a-}\Gamma^{a+} = 1 - \Gamma^{a+}\Gamma^{a-} \quad \text{if } s_a = \frac{1}{2} \end{cases} \tag{21.1.11}$$

Using (21.1.11) in (21.1.10), we have;

$$S_a\zeta^{\mathsf{s}} = \begin{cases} -\frac{1}{2}\zeta^{\mathsf{s}} \text{ if } s_a = -\frac{1}{2} \\ \zeta^{\mathsf{s}} - \frac{1}{2}\zeta^{\mathsf{s}} = \frac{1}{2}\zeta^{\mathsf{s}} \text{ if } s_a = \frac{1}{2} \end{cases} \Rightarrow S_a\zeta^{\mathsf{s}} = s_a\zeta^{\mathsf{s}}$$

where we used the fact that $\Gamma^{a-}\zeta = 0$. So, ζ^{s} is a simultaneous eigenstate of S_a. It also means that ζ^{s} with even/odd number of non-zero s_a's don't mix.
We now define the chirality matrix Γ as follows;

$$\Gamma = i^{-k}\Gamma^0...\Gamma^{d-1} \tag{21.1.12}$$

This matrix has some useful properties that we now prove.

$$\Gamma^2 = i^{-2k}\Gamma^0\Gamma^1...\Gamma^{d-1}\Gamma^0\Gamma^1...\Gamma^{d-1} = (-1)^k[(-1)^{d-1}(-1)](-1)^{d-2}...(-1)^1(-1)^0$$

$$= (-1)^P \text{ where } P = k + 1 + \sum_{j=0}^{d-1} j = k + 1 + \frac{(d-1)d}{2} = k + 1 + \frac{(2k+1)(2k+2)}{2} = 2(k+1)^2$$

$$\Rightarrow (-1)^P = 1 \Rightarrow \Gamma^2 = 1 \tag{21.1.13}$$

where the powers of -1 in the first line of (21.1.13) comes as follows. The d−2 power comes due to anti-commuting Γ^0 all the way through to other Γ^0 and the other -1 in the square bracket comes because Γ^0 squares to -1. The rest of the powers after that come due to anti-commuting gamma matrices. We now prove another property;

$$\{\Gamma, \Gamma^\mu\} = i^{-k}\{\Gamma^0...\Gamma^{d-1}, \Gamma^\mu\} = i^{-k}\left(\Gamma^0...\Gamma^\mu...\Gamma^{d-1}\Gamma^\mu + \Gamma^\mu\Gamma^0...\Gamma^\mu...\Gamma^{d-1}\right)$$

$$= i^{-k}\left((-1)^{d-\mu-1}\Gamma^0...\Gamma^{\mu-1}\Gamma^{\mu+1}...\Gamma^{d-1} + (-1)^\mu\Gamma^0...\Gamma^{\mu-1}\Gamma^{\mu+1}...\Gamma^{d-1}\right)$$

$$= i^{-k}\left((-1)^{d-\mu-1} + (-1)^\mu\right)\Gamma^0...\Gamma^{\mu-1}\Gamma^{\mu+1}...\Gamma^{d-1} = i^{-k}\left((-1)^{d+\mu-1} + (-1)^\mu\right)\Gamma^0...\Gamma^{\mu-1}\Gamma^{\mu+1}...\Gamma^{d-1}$$

$$= i^{-k}(-1)^\mu\left((-1)^{d-1} + 1\right)\Gamma^0...\Gamma^{\mu-1}\Gamma^{\mu+1}...\Gamma^{d-1} = 0$$

where in the last step, we used the fact that $(-1)^{d-1} = -1$ because d is even. So, we have shown that;

$$[\Gamma, \Gamma^\mu] = 0 \tag{21.1.14}$$

This property readily implies the following property;

$$[\Gamma,\Sigma^{\mu\nu}] = -\frac{i}{4}[\Gamma,[\Gamma^{\mu},\Gamma^{\nu}]] = -\frac{i}{4}[\Gamma[\Gamma^{\mu},\Gamma^{\nu}]-[\Gamma^{\mu},\Gamma^{\nu}]\Gamma] = -\frac{i}{4}[\Gamma\Gamma^{\mu}\Gamma^{\nu}-\Gamma\Gamma^{\nu}\Gamma^{\mu}-\Gamma^{\mu}\Gamma^{\nu}\Gamma+\Gamma^{\nu}\Gamma^{\mu}\Gamma]$$
$$= -\frac{i}{4}[\Gamma\Gamma^{\mu}\Gamma^{\nu}-\Gamma\Gamma^{\nu}\Gamma^{\mu}-\Gamma\Gamma^{\mu}\Gamma^{\nu}+\Gamma\Gamma^{\nu}\Gamma^{\mu}] = 0 \tag{21.1.15}$$

In the last step, we used (21.1.14). Final property is derived as follows;

$$\Gamma = i^{-k}\Gamma^{0}...\Gamma^{d-1} = 2^{k+1}\left(\frac{1}{2}\Gamma^{0}\Gamma^{1}\right)\left(-\frac{i}{2}\Gamma^{2}\Gamma^{3}\right)...\left(-\frac{i}{2}\Gamma^{2k}\Gamma^{2k+1}\right) = 2^{k+1}S_0...S_k \tag{21.1.16}$$

which implies that Γ is diagonal and it is easy to see that if even number of s_a's are $1/2$, then the diagonal entry is $1/2$ and if odd number of s_a's are $1/2$, then the diagonal entry is $-1/2$. Thus, the representation breaks into two representations with chirality +1 and −1 respectively. For odd dimensions i.e. $d = 2k+3$, add Γ as an additional gamma matrix i.e. $\Gamma^{2k+3} = \Gamma$ and we get an irreducible representation of Lorentz algebra.

21.1.1 Majorana Spinors

Let s be the basis where the matrix elements of Γ^{a+} are real. This implies that $\Gamma^{3},...\Gamma^{d-1} = \Gamma^{2k+1}$ are purely imaginary and $\Gamma^{0},\Gamma^{1},\Gamma^{2},\Gamma^{4}...\Gamma^{d} = \Gamma^{2k+2}$ are real. Moreover, since $\Gamma^{\mu},\Gamma^{*\mu}$ and $-\Gamma^{*\mu}$ satisfy the Clifford algebra, they should be related by a similarity transformation. We will find this similarity transformation. We define two matrices as follows;

$$B_1 = \Gamma^{3}..\Gamma^{2k+1},\ \ B_2 = \Gamma B_1 \tag{21.1.17}$$

We now derive the following useful property. For even μ, we have;

$$B_1\Gamma^{\mu}B_1^{-1} = \Gamma^{3}..\Gamma^{2k+1}\Gamma^{\mu}\Gamma^{2k+1}..\Gamma^{3} = (-1)^{k}\Gamma^{\mu} = (-1)^{k}\Gamma^{*\mu}$$

Moreover, for odd μ, we have;

$$B_1\Gamma^{\mu}B_1^{-1} = B_1\Gamma^{2\nu+1}B_1^{-1} = \Gamma^{3}..\Gamma^{2\nu+1}...\Gamma^{2k+1}\Gamma^{\mu}\Gamma^{2k+1}..\Gamma^{2\nu+1}...\Gamma^{3}$$
$$= (-1)^{k-\nu}\Gamma^{3}..\Gamma^{2\nu-1}\Gamma^{2\nu+3}...\Gamma^{2k+1}\Gamma^{2k+1}..\Gamma^{2\nu+3}\Gamma^{2\nu+1}\Gamma^{2\nu-1}...\Gamma^{3}$$
$$= (-1)^{k-\nu}(-1)^{\nu-1}\Gamma^{3}..\Gamma^{2\nu-1}\Gamma^{2\nu+3}...\Gamma^{2k+1}\Gamma^{2k+1}..\Gamma^{2\nu+3}\Gamma^{2\nu-1}...\Gamma^{3} = (-1)^{k}(-\Gamma^{\mu}) = (-1)^{k}\Gamma^{*\mu}$$

So, we see that for all μ, we have;

$$B_1\Gamma^{\mu}B_1^{-1} = (-1)^{k}\Gamma^{*\mu} \tag{21.1.18}$$

A similar property for B_2 is derived as follows;

$$B_2\Gamma^{\mu}B_2^{-1} = \Gamma B_1\Gamma^{\mu}B_1^{-1}\Gamma = (-1)^{k}\Gamma\Gamma^{*\mu}\Gamma = (-1)^{k+1}\Gamma^{*\mu}\Gamma^{2} = (-1)^{k+1}\Gamma^{*\mu} \tag{21.1.19}$$

Anther property for B_1 and B_2 is derived as follows;

$$B_1\Sigma^{\mu\nu}B_1^{-1} = -\frac{i}{4}B_1[\Gamma^{\mu},\Gamma^{\nu}]B_1^{-1} = -\frac{i}{4}(B_1\Gamma^{\mu}B_1^{-1}B_1\Gamma^{\nu}B_1^{-1}-B_1\Gamma^{\nu}B_1^{-1}B_1\Gamma^{\mu}B_1^{-1}) = -(-1)^{2k}\frac{i}{4}[\Gamma^{\mu},\Gamma^{\nu}] = -\Sigma^{*\mu\nu}$$
$$B_2\Sigma^{\mu\nu}B_2^{-1} = -\frac{i}{4}B_2[\Gamma^{\mu},\Gamma^{\nu}]B_2^{-1} = -\frac{i}{4}(B_2\Gamma^{\mu}B_2^{-1}B_2\Gamma^{\nu}B_2^{-1}-B_2\Gamma^{\nu}B_2^{-1}B_2\Gamma^{\mu}B_2^{-1}) = -(-1)^{2k+2}\frac{i}{4}[\Gamma^{\mu},\Gamma^{\nu}] = -\Sigma^{*\mu\nu} \tag{21.1.20}$$

We see that ζ and $B^{-1}\zeta^{*}$ transforms in the same way under lorentz transformations;

$$\zeta \to (1+i\omega_{\mu\nu}\Sigma^{\mu\nu})\zeta \tag{21.1.21}$$

$$\Rightarrow B^{-1}\zeta^{*} \to B^{-1}(1-i\omega_{\mu\nu}\Sigma^{*\mu\nu})\zeta^{*} = B^{-1}\zeta^{*}-i\omega_{\mu\nu}B^{-1}\Sigma^{*\mu\nu}BB^{-1}\zeta^{*} = B^{-1}\zeta^{*}+i\omega_{\mu\nu}\Sigma^{\mu\nu}B^{-1}\zeta^{*} = (1+i\omega_{\mu\nu}\Sigma^{\mu\nu})B^{-1}\zeta^{*} \tag{21.1.22}$$

where B stands for B_1 and B_2 collectively and $\omega_{\mu\nu}$ are real parameters. So, Dirac representation is self-conjugate. Now, we prove the following identities;

$$B_1\Gamma B_1^{-1} = (i)^{-k}\Gamma^3\Gamma^5...\Gamma^{d-1}\left(\Gamma^0\Gamma^1...\Gamma^{d-1}\right)\Gamma^{d-1}...\Gamma^3 = (i)^{-k}(-1)^{k(d-1)}\Gamma^0\Gamma^1...\Gamma^{d-1} = i^k\Gamma^0\Gamma^1...\Gamma^{d-1}$$

$$= i^k(-1)^k\Gamma^0\Gamma^1\Gamma^2\Gamma^{*3}...\Gamma^{*(d-1)} = (-1)^k\Gamma^*$$

Before proving the next identity, we notice that $B_1\Gamma B_1^{-1}$ is also $i^{2k}\Gamma$ and thus, $i^{2k}\Gamma = (-1)^k\Gamma^*$. The second identity is as follows;

$$B_2\Gamma B_2^{-1} = \Gamma B_1\Gamma B_1^{-1}\Gamma = i^{2k}\Gamma\Gamma\Gamma = (-1)^k\Gamma = (-1)^k\Gamma^*$$

Now, suppose that ξ is a spinor whose chirality is α i.e.;

$$\Gamma\xi = \alpha\xi$$

then, the chirality of the conjugate representation is as follows;

$$\Gamma(B_1^{-1}\xi^*) = B_1^{-1}B_1\Gamma B_1^{-1}\xi^* = (-1)^k B_1^{-1}(\Gamma\xi)^* = \alpha(-1)^k B_1^{-1}\xi^*$$

So, the chirality of a spinor is changed if k is odd (which corresponds to $d = 0$ mod 4 e.g. $d = 4, 8$) and it doesn't change if k is even (which corresponds to $d = 2$ mod 4 e.g. $d = 2, 6, 10$). Thus, Weyl representation is self conjugate for $d = 2$ mod 4 and it is not self conjugate for $d = 0$ mod 4. Now, we want to impose a Majorana condition i.e. a condition that relates a spinor ζ to it's conjugate ζ^*. Such a condition can be written as follows;

$$\zeta^* = B\zeta \tag{21.1.23}$$

where $B = B_1$ or B_2. We see that (21.1.23) is consistent with Lorentz transformations using (21.1.21) by showing that both sides of (21.1.23) transform in the same way;

$$\zeta^* \to (1 - i\omega_{\mu\nu}\Sigma^{*\mu\nu})\zeta^*$$

$$B\zeta \to B(1 + i\omega_{\mu\nu}\Sigma^{\mu\nu})\zeta = B\zeta + i\omega_{\mu\nu}B\Sigma^{\mu\nu}B^{-1}B\zeta = B\zeta - i\omega_{\mu\nu}\Sigma^{*\mu\nu}B\zeta = (1 - i\omega_{\mu\nu}\Sigma^{*\mu\nu})B\zeta$$

Taking the conjugate of (21.1.23), we get;

$$\zeta = B^*\zeta^* = B^*B\zeta \Rightarrow B^*B = 1 \tag{21.1.24}$$

Now, we see that;

$$B_1^*B_1 = \Gamma^{*3}\Gamma^{*5}...\Gamma^{*(d-1)}\Gamma^3\Gamma^5...\Gamma^{d-1} = (-1)^k\Gamma^3\Gamma^5...\Gamma^{d-1}\Gamma^3\Gamma^5...\Gamma^{d-1} = (-1)^k(-1)^{k-1}(-1)^{k-2}...(-1)^0$$

$$= (-1)^{k+(k-1)+(k-2)+...+1} = (-1)^{k+\frac{k(k-1)}{2}} = (-1)^{\frac{k(k+1)}{2}}$$

Similarly, we see that (do this);

$$B_2^*B_2 = \Gamma^*B_1^*\Gamma B_1 = (-1)^{\frac{k(k-1)}{2}}$$

So, B_1 can be used for Majorana condition only if;

$$\frac{k(k+1)}{2} = 2n \;\; n \in \mathbb{Z}_0^+$$

$$\Rightarrow k(k+1) = 4n$$

Since k and k+1 differ by 1, one of them have to be odd and thus, either k is a multiple of 4 or $k+1$ is a multiple of 4. So, we get;

$$k = 0 \text{ mod } 4 \text{ or } k = 3 \text{ mod } 4$$

$$\Rightarrow d = 2 \text{ mod } 8 \text{ or } d = 0 \text{ mod } 8 \text{ (for } B_1) \tag{21.1.25}$$

Similarly, B_2 can be used for Majorana condition only if;

$$\frac{k(k-1)}{2} = 2n \;\; n \in \mathbb{Z}_0^+$$

$$\Rightarrow k(k-1) = 4n$$

Since k and $k-1$ differ by 1, one of them have to be odd and thus, either k is a multiple of 4 or $k-1$ is a multiple of 4. So, we get;

$$k = 0 \text{ mod } 4 \text{ or } k = 1 \text{ mod } 4$$

$$\Rightarrow d = 2 \text{ mod } 8 \text{ or } d = 4 \text{ mod } 8 \text{ (for } B_2\text{)} \tag{21.1.26}$$

Now, (21.1.23) implies $\zeta = B^{-1}\zeta^*$ and thus, if we want to impose Majorana condition on a Weyl spinor, then the Weyl spinor have to be self conjugate. Thus, the condition of Weyl self-conjugacy (i.e. $d = 2$ mod 4) have to be consistent with either (21.1.25) or (21.1.26). We see that the only case that is consistent with Weyl self-conjugacy is $d = 2$ mod 8. Since $\Gamma^{\mu T}$ and $-\Gamma^{\mu T}$ satisfy the Clifford algebra as well, they should be related to Γ^μ by a similarity transformation. We will now find that similarity transformation. Define the charge conjugation matrix as follows;

$$C\Gamma^\mu C^{-1} = -\Gamma^{\mu T} \tag{21.1.27}$$

this implies the following;

$$(C\Gamma^0)\Gamma^\mu(C\Gamma^0)^{-1} = \Gamma^{*\mu} \tag{21.1.28}$$

Comparing (21.1.28) with (21.1.18) and (21.1.19), we see that;

$$C\Gamma^0 = B_1 \Rightarrow C = -B_1\Gamma^0 \quad \text{if } k = 0 \text{ mod } 2 \Rightarrow d = 2 \text{ mod } 4$$

$$C\Gamma^0 = B_2 \Rightarrow C = -B_2\Gamma^0 \quad \text{if } k+1 = 0 \text{ mod } 2 \Rightarrow d = 0 \text{ mod } 4$$

We also derive a useful property for C as follows;

$$C\Sigma^{\mu\nu}C^{-1} = -\frac{i}{4}(C\Gamma^\mu C^{-1}C\Gamma^\nu C^{-1} - C\Gamma^\nu C^{-1}C\Gamma^\mu C^{-1}) = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]^T = -\Sigma^{\mu\nu T} \tag{21.1.29}$$

21.1.2 Product representations

We first prove that the spinor $\bar{\xi}$ and the spinor $\xi^T C$ transform the same way as follows;

$$\xi \to (1 + i\omega_{\mu\nu}\Sigma^{\mu\nu})\xi$$

$$\Rightarrow \bar{\xi} \to \bar{\xi}(1 + i\omega_{\mu\nu}\Gamma^0\Sigma^{\dagger\mu\nu}\Gamma^0) \tag{21.1.30}$$

To continue, we calculate the following

$$\Gamma^0\Sigma^{\dagger\mu\nu}\Gamma^0 = \frac{i}{4}\Gamma^0(\Gamma^0\Gamma^\nu\Gamma^0\Gamma^0\Gamma^\mu\Gamma^0 - \Gamma^0\Gamma^\mu\Gamma^0\Gamma^0\Gamma^\nu\Gamma^0)\Gamma^0 = -\Sigma^{\mu\nu}$$

So, (21.1.30) becomes;

$$\bar{\xi} \to \bar{\xi}(1 - i\omega_{\mu\nu}\Sigma^{\mu\nu}) \tag{21.1.31}$$

Moreover, we see that;

$$\xi^T C \to (1 + i\omega_{\mu\nu}\Sigma^{\mu\nu T})C = \xi^T C(1 + i\omega_{\mu\nu}C^{-1}\Sigma^{\mu\nu T}C) = \xi^T C(1 - i\omega_{\mu\nu}\Sigma^{\mu\nu}) \tag{21.1.32}$$

where in the last step, we used (21.1.29).Comparing (21.1.31) with (21.1.31), we see that the spinor $\bar{\xi}$ and the spinor $\xi^T C$ transform the same way.
We now prove the following identity for even d (Prove this);

$$\Gamma^{\mu_1\cdots\mu_s}\Gamma = \frac{i^{-k+s(s-1)}}{(d-s)!}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d} \quad s < d \tag{21.1.33}$$

which implies that in odd dimensions (where $\Gamma^d = \Gamma$), say d+1 dimensions, (where d is the same as in (21.1.33)), an $s+1$ form is related to $d-s$ form which is the same as $d+1-(s+1)$ form. Now, let $d = 2k$+3 be the number of odd dimensions. Then, the independent forms have the following number of indices;

$$0, 1, ...\frac{d-1}{2} \quad \text{or } 0, 1, ...k+1$$

This leads us to the following forms;

$$[0], [1], ...[k+1]$$

$$\Rightarrow 2^{\mathtt{k+1}} \times 2^{\mathtt{k+1}} = [0] + [1] + ... + [k+1] \quad \text{odd } d \tag{21.1.34}$$

where the square brackets indicate the corresponding form. In even dimensions, the linear relation mentioned above doesn't exist and thus, the allowed forms are as follows;

$$[0], [1], ...[d] = [0], [1], ...[2k+2]$$

But since $[m]$ can be related to $[d-m]$ via Γ, we get the following forms ($[k$+1$]$ is related to itself and hence, we count it only once).

$$[0]^2, [1]^2, ...[k]^2, [k+1]$$

$$\Rightarrow 2^{\mathtt{k+1}}_{\mathtt{Dirac}} \times 2^{\mathtt{k+1}}_{\mathtt{Dirac}} = [0]^2, [1]^2 + ... + [k]^2 + [k+1] \quad \text{even } d \tag{21.1.35}$$

Now, we prove the following identity;

$$\zeta^T C \Gamma^{\mu_1 \mu_2 ... \mu_m} \Gamma \chi = (-1)^{k+m+1} (\Gamma\zeta)^T C \Gamma^{\mu_1 ... \mu_m} \chi \tag{21.1.36}$$

where ζ and χ are arbitrary spinors. The RHS is manipulated as follows;

$$\zeta^T C \Gamma^{\mu_1 \mu_2 ... \mu_m} \Gamma \chi = \frac{(-1)^m}{m!} \delta^{\mu_1 ... \mu_m}_{\nu_1 ... \nu_m} \zeta^T C \Gamma \Gamma^{\mu_1 \mu_2 ... \mu_m} \chi = \frac{(-1)^{k+m+1}}{m!} \delta^{\mu_1 ... \mu_m}_{\nu_1 ... \nu_m} (\Gamma\zeta)^T C \Gamma^{\mu_1 \mu_2 ... \mu_m} \chi$$

$$= (-1)^{k+m+1} (\Gamma\zeta)^T C \Gamma^{\mu_1 \mu_2 ... \mu_m} \chi \text{ (Shown)}$$

where I used the following identity;

$$\zeta^T C \Gamma = \zeta^T i^{-k} C \Gamma^0 ... \Gamma^{d-1} = \zeta^T i^{-k} C \Gamma^0 C^{-1} ... C \Gamma^{d-1} C^{-1} C = \zeta^T i^{-k} (-1)^d \Gamma^{0T} ... \Gamma^{(d-1)T} C = (i^{-k} \Gamma^{d-1} ... \Gamma^0 \zeta)^T C$$

$$= (-1)^{d-1} ... (-1)^1 (i^{-k} \Gamma^0 ... \Gamma^{d-1} \zeta)^T C = (-1)^{k+1} (\Gamma\zeta)^T C$$

So, we see that $\zeta^T C \Gamma^{\mu_1 \mu_2 ... \mu_m} \chi$ is non vanishing only if k+m is odd and ζ, χ have the same chirality or when $k+m$ is even and ζ, χ have the opposite chirality. So, we get the following;

$$2^{\mathtt{k}}_{\mathtt{Weyl}} \times 2^{\mathtt{k}}_{\mathtt{Weyl}} = \begin{cases} [1] + [3] + ...[k+1]_+ & \text{even } k \\ [0] + [2] + ...[k+1]_+ & \text{odd } k \end{cases}$$

$$2^{\mathtt{k}'}_{\mathtt{Weyl}} \times 2^{\mathtt{k}'}_{\mathtt{Weyl}} = \begin{cases} [1] + [3] + ...[k+1]_- & \text{even } k \\ [0] + [2] + ...[k+1]_- & \text{odd } k \end{cases}$$

$$2^{\mathtt{k}}_{\mathtt{Weyl}} \times 2^{\mathtt{k}'}_{\mathtt{Weyl}} = \begin{cases} [0] + [2] + ...[k] & \text{even } k \\ [1] + [3] + ...[k] & \text{odd } k \end{cases} \tag{21.1.37}$$

where the subscripts on $[k$+1$]$ are there because k+1 is related to itself by (21.1.33) and thus, it has a self dual part and an anti self dual part. (True?).
In the end of the section, Polchinski re-derives some facts using s_a eigenvalues. We prove the following identities first.

$$S_0 \zeta = s_0 \zeta \Rightarrow S_0 B^{-1} \zeta^* = s_0 B^{-1} \zeta^*, \quad S_a \zeta = s_a \zeta \Rightarrow S_a B^{-1} \zeta^* = -s_a B^{-1} \zeta^* \quad a = 1, ...k \tag{21.1.38}$$

The first identity is derived as follows;

$$S_0 B^{-1}\zeta^* = \frac{1}{2}\Gamma^0\Gamma^1 B^{-1}\zeta^* = \frac{1}{2}B^{-1}B\Gamma^0 B^{-1}B\Gamma^1 B^{-1}\zeta^* = \frac{1}{2}B^{-1}\Gamma^{*0}\Gamma^{*1}\zeta^* = B^{-1}(S_0\zeta)^* = s_0 B^{-1}\zeta^*$$

where the last line, I used the fact that s_0 has to be real (because Γ^0, Γ^1 and Weyl spinors are real). The second identity is similarly proven as follows;

$$S_a B^{-1}\zeta^* = \frac{-i}{2}\Gamma^{2a}\Gamma^{2a+1}B^{-1}\zeta^* = \frac{-i}{2}B^{-1}B\Gamma^{2a}B^{-1}B\Gamma^{2a+1}B^{-1}\zeta^* = \frac{-i}{2}B^{-1}\Gamma^{*2a}\Gamma^{*2a+1}\zeta^* = -B^{-1}(S_a\zeta)^* = -s_a B^{-1}\zeta^*$$

So, under conjugation, $s_1, ..., s_k$ are flipped. For even k, this is an even number of flips and hence, the chirality is unchanged (see (21.1.16)). For odd k, this is an odd number of flips and hence, the chirality changes. Thus, self-conjugate Weyl reps can be in even k (or for $d = 2$ mod 4) only as we derived before.

21.1.3 Spinors of $SO(N)$

The results for $SO(N)$ for $N = 2l$ are like $SO(N+1, 1)$ (write more about this). The breakdowns are as follows;

$$\mathbf{2^{l-1}} \times \mathbf{2^{l-1}} = \begin{cases} [0] + [2] + ...[l]_+ & \text{even } k \\ [1] + [3] + ...[l]_+ & \text{odd } k \end{cases}$$

$$\mathbf{2^{l-1\prime}} \times \mathbf{2^{l-1\prime}} = \begin{cases} [0] + [2] + ...[l]_- & \text{even } k \\ [1] + [3] + ...[l]_- & \text{odd } k \end{cases}$$

$$\mathbf{2^{l-1}} \times \mathbf{2^{l-1\prime}} = \begin{cases} [1] + [3] + ...[l-1] & \text{even } k \\ [0] + [2] + ...[l-1] & \text{odd } k \end{cases} \tag{21.1.39}$$

21.1.4 Decomposition under subgroups

If we consider the following decomposition;

$$SO(2k+1) \to SO(2l+1) \times SO(2k-2l)$$

then the Weyl representation $\mathbf{2^k}$ of $SO(2k+1, 1)$ decomposes as follows (derive);

$$\mathbf{2^k} \to (\mathbf{2^l}, \mathbf{2^{k-l-1}}) \oplus (\mathbf{2'^l}, \mathbf{2'^{k-l-1}}) \tag{21.1.40}$$

Derive the SO(2N) decomposition to SU(N)

21.2 Introduction to supersymmetry: $d = 4$

Most of the results in this section are presented without derivation in the original text. For more details, we can consult sources such as [1] for a short introduction or [2] for a more extended introduction. I will try to justify as many missing steps from Polchinski's original text as possible to make sense of the discussion.

21.2.1 $d = 4, N = 1$ supersymmetry

IN $d = 4$, the smallest spinor representation has $2^{4/2} = 4$ degrees of freedom. Since 4 can't be written as $2k+8$ with k a non-negative integer, we can't place a Majorana and Weyl condition on the spinor simultaneously. The Majorana condition can be placed but the Weyl representation is not self-conjugate. So, the four degrees of freedom can be realized in one of the following ways;

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \quad \psi^1, \psi^2 \in \mathbb{C} \quad \text{or} \quad \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} \quad \psi^1, \psi^2, \psi^3, \psi^4 \in \mathbb{R}$$

The first way corresponds to a Weyl spinor and the second corresponds to a Majorana spinor. In the text, the second way (i.e. Majorana) is used. So, the supercharges are denoted as Q_α with $\alpha \in \{0,1,2,3\}$ and due to the Majorana condition, we have;

$$\bar{Q}_\alpha = Q_\alpha^\dagger \Gamma^0 = (Q_\alpha^*)^T \Gamma^0 = (B_1 Q_\alpha)^T \Gamma^0 = Q_\alpha^T B_1^T \Gamma^0 = Q_\alpha^T C$$

(justify the last step.) where we used B_1 for the Majorana condition because $4 = 0 \pmod 4$. Now, the supersymmetry algebra for using only one Majorana spinor (which is $N = 1$ supersymmetry with 4 supercharges) contains the following important commutators and anticommutators;

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \Gamma^\mu_{\alpha\beta} \tag{21.2.1}$$

$$[P^\mu, Q_\alpha] = 0 \tag{21.2.2}$$

(In some resources like [2] and [1], the minus sign won't be there in the $\{Q, \bar{Q}\}$ anticommutator because they are using the $(+1,-1,-1,-1)$ convention for the metric. They are also using the Weyl spinors for SUSY instead of the Majorana spinors. So, there will be a difference due to that as well). Since $4 = 2(1)+2$, we have $k = 1$ in this case (in the formalism of the previous section) and thus, the spinors can be labeled by the (s_0, s_1) 2-tuple. We can now work out the representations of $(21.2.1)$.

If the representation is massless, then the P_μ vector can be written as follows;

$$P_\mu = (-k^0, k^0, 0, 0) \Rightarrow -2P_\mu\Gamma^\mu = -2(-k^0\Gamma^0 + k^0\Gamma^1) = 2k_0(\Gamma^0 - \Gamma^1) = 2k^0\Gamma^0(-1 - \Gamma^0\Gamma^1)$$

$$= 2k^0\Gamma^0(1 + \Gamma^0\Gamma^1) = 2k^0\Gamma^0(1 + 2S_0)$$

where we used the definition of S_0 from previous section and the fact that $(\Gamma^0)^2 = -1$. So, $(21.2.1)$ becomes;

$$\{Q_\alpha, \bar{Q}_\beta\} = 2k^0\Gamma^0_{\alpha\gamma}(1 + 2S_0)_{\gamma\beta}$$

$$\Rightarrow Q_\alpha\bar{Q}_\beta + \bar{Q}_\beta Q_\alpha = 2k^0\Gamma^0_{\alpha\gamma}(1 + 2S_0)_{\gamma\beta} \Rightarrow Q_\alpha Q^\dagger_\gamma \Gamma^0_{\gamma\beta} + Q^\dagger_\gamma \Gamma^0_{\gamma\beta} Q_\alpha = 2k^0\Gamma^0_{\alpha\gamma}(1 + 2S_0)_{\gamma\beta}$$

$$\Rightarrow Q_\alpha Q^\dagger_\beta + Q^\dagger_\beta Q_\alpha = 2k^0(1 + 2S_0)_{\alpha\beta} \Rightarrow \{Q_\alpha, Q^\dagger_\beta\} = 2k^0(1 + 2S_0)_{\alpha\beta}$$

(Justify the second last step). So, we see that the anti-commutator vanishes if $s_0 = -1/2$. This means that (labeling the spinors by the (s_0, s_1) tuple);

$$||Q_{-1/2,s_1}|\psi\rangle||^2 + ||Q^\dagger_{-1/2,s_1}|\psi\rangle||^2 = \langle\psi|Q^\dagger_{-1/2,s_1}Q_{-1/2,s_1}|\psi\rangle + \langle\psi|Q_{-1/2,s_1}Q^\dagger_{-1/2,s_1}|\psi\rangle = \langle\psi|\{Q_{-1/2,s_1}, Q^\dagger_{-1/2,s_1}\}|\psi\rangle = 0$$

where $|\psi\rangle$ is any state in the Hilbert space. Due to the positivity of the Hilbert space, we then require that $Q_{-1/2,s_1}$ vanish identically. From $(21.1.38)$, we see that if

$$S_0 Q = s_0 Q, \quad S_1 Q = s_1 Q$$

then,

$$S_0 B^{-1} Q^* = s_0 B^{-1} Q^*, \quad S_1 B^{-1} Q^* = -s_1 B^{-1} Q^*$$

So, the conjugation flips the s_1 eigenvalue. Thus, the Majorana condition becomes;

$$Q^\dagger_{s_0,s_1} = Q_{s_0,-s_1}$$

(Justify this). In the s basis, the anticommutator becomes;

$$\{Q_{s_0', s_1'}, Q^\dagger_{s_0,s_1}\} = 4k^0\delta_{s_0,1/2}\delta_{\mathtt{s},\mathtt{s}'}$$

where we used the fact that this anti-commutator is zero if $s_0 = -1/2$ and thus, we put in $S_0 = 1/2$ in the expression of the anti-commutator to get a factor of 4. Moreover, $s_1' = s_1$ because if $s_1' = -s_1$ then

$$Q^\dagger_{1/2,s_1} = Q^\dagger_{1/2,-s_1'} = Q_{1/2,s_1'}$$

and thus, the anticommutator vanishes because we will have terms like $Q^2_{1/2,s_1'}$. Therefore, we have a factor of $\delta_{\mathtt{s},\mathtt{s}'}$ in the anticommutator. The nonzero anticommutator is thus;

$$\{Q_{1/2,-1/2}, Q^\dagger_{1/2,-1/2}\} = \{Q_{1/2,-1/2}, Q_{1/2,1/2}\} = 4k^0 \Rightarrow \{b, b^\dagger\} = 1$$

where

$$b = \frac{1}{2\sqrt{k^0}} Q_{1/2,-1/2},\ b^\dagger = \frac{1}{2\sqrt{k^0}} Q_{1/2,1/2}$$

This also implies

$$b^2 = b^{\dagger 2} = 0$$

which gives us a set of two states starting from any state (called the Clifford vaccum);

$$S_1|\Omega\rangle = \Omega|\Omega\rangle;\ \ b|\Omega\rangle = \left|\Omega + \frac{1}{2}\right\rangle$$

where the state is labelled by its S_1 eigenvalue. It can be seen from the definition of S_1 that S_1 is helicity (justify this more). The relevant massless representations are as follows;

- $\Omega = 0$: We have the following multiplet

$$\left(|0\rangle + \left|\frac{1}{2}\right\rangle\right) \oplus \left(\left|-\frac{1}{2}\right\rangle + |0\rangle\right)$$

  This is called the chiral multiplet.

21.2.2 Actions with $d = 4, N = 1$ SUSY

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21.2.3 Spontaneous symmetry breaking

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21.2.4 Higher corrections and supergravity

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21.2.5 Extended supersymmetry in $d = 4$

We can have more than one Majorana spinor containing the supercharges. We labvel these spinors by an index $N = 1, 2, \ldots$. The generalization (but not the most general) of the SUSY algebra in (21.2.1) is as follows;

$$\{Q^A_\alpha, \bar{Q}^A_\alpha\} = -2\delta^{AB} P_\mu \Gamma^\mu \alpha\beta$$

with P_μ still commuting with all of Q^A_α's. In parallel to the calculation before, in the (s_0, s_1) basis, we have;

$$\{Q^A_{s_0', s_1'}, \bar{Q}^B_{s_0,s_1}\} = 4k^0 \delta^{AB} \delta_{s_0,1/2} \delta_{\mathtt{s},\mathtt{s}'}$$

and thus, we now have N different b^A oscillators. All of them square to one and thus, starting with the $|\Omega\rangle$ state, we can build states that are now labeled by the occupancy numbers of these oscillators $(n_1, .., n_N)$ which can be 0 or 1. Therefore, there are 2^N different states. With k occupancy numbers non-zero, the state helicity is $\lambda + k/2$. For example, for $N = 2$ the states are as follows;

$$\underbrace{|\lambda\rangle}_{\texttt{helicity } \lambda} \quad \underbrace{b^1|\lambda\rangle,\ b^2|\lambda\rangle}_{\texttt{helicity } \lambda+1/2} \quad \underbrace{b^1 b^2|\lambda\rangle}_{\texttt{helicity } \lambda+1}$$

This gives us the following multiples;

- $\lambda = -1/2$: The multiplets are as follows;

$$\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

  The CPT conjugate has the same helicities as the original multiplet. This multiplet is called a hypermultiplet. Justify the use of CPT... scalars not in correct representation

- $\lambda = 0$: The multiplets are as follows;

$$\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$$

  This multiplet contains the vector excitation and thus, it is the vectror multiplet.

- $\lambda = 1$: The multiplets are as follows;

$$\left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \oplus \left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$$

  This multiplet contains the graviton and thus, it is the supergravity multiplet.

There would be a multiplet with $\lambda = 3/2$ as well but we didn't discuss it. In hypermultiplet, both fermions are in the same multiplet and thus, they should be in the same gauge group. In the vector multiplet, fermions are in the same multiplet as the vector and thus, they should be in the adjoint representation (elaborate on this a little).
For $N = 4$ case, the states are as follows;

$$|\lambda\rangle \;\; 1 \text{ state, helicity } \lambda$$

$$b^A|\lambda\rangle \;\; \binom{4}{1} = 4 \text{ states, helicity } \lambda + \frac{1}{2}$$

$$b^A b^B|\lambda\rangle \;\; \binom{4}{2} = 6 \text{ states, helicity } \lambda + 1$$

$$b^A b^B b^C|\lambda\rangle \;\; \binom{4}{3} = 4 \text{ states, helicity } \lambda + \frac{3}{2}$$

$$b^1 b^2 b^3 b^4|\lambda\rangle \;\; 1 \text{ state, helicity } \lambda + 2$$

The relevant multiplets are as follows;

- $\lambda = -1$: The multiplet is as follows;

$$\left(-1, -\frac{1}{2}^4, 0^6, \frac{1}{2}^4, 1\right)$$

  This multiplet contains the vector but not the graviton and thus, it is the vector multiplet. The exponents indicate the number of times a state with the indicated helicity appears in the multiplet. The CPT conjugate is not required because the scalars live in $SO(6)$ (elaborate on that).

- $\lambda = 0$: The multiplets are as follows;

$$\left(-2, -\frac{3}{2}^4, 1^6, -\frac{1}{2}^4, 0\right) \oplus \left(0, \frac{1}{2}^4, 1^6, \frac{3}{2}^4, 2\right)$$

  This multiplet contains the vector but not the graviton and thus, it is the vector multiplet. We can't have $\lambda > 0$ because that gives helicities greater than 2 and this is not allowed due to no-go theorems (see for example, Weinberg-Witten theorem).

For $N = 8$ case, the states are as follows;

$$|\lambda\rangle \;\; 1 \text{ state, helicity } \lambda$$

$$b^A|\lambda\rangle \;\; \binom{8}{1} = 8 \text{ states, helicity } \lambda + \frac{1}{2}$$

$$b^A b^B|\lambda\rangle \;\; \binom{8}{2} = 28 \text{ states, helicity } \lambda + 1$$

$$b^A b^B b^C|\lambda\rangle \;\; \binom{8}{3} = 56 \text{ states, helicity } \lambda + \frac{3}{2}$$

$$b^A b^B b^C b^D|\lambda\rangle \quad \binom{8}{4} = 70 \text{ states, helicity } \lambda+2$$

$$b^A b^B b^C b^D b^E|\lambda\rangle \quad \binom{8}{5} = 56 \text{ states, helicity } \lambda+\frac{5}{2}$$

$$b^A b^B b^C b^D b^E b^F|\lambda\rangle \quad \binom{8}{6} = 28 \text{ states, helicity } \lambda+3$$

$$b^A b^B b^C b^D b^E b^F b^G|\lambda\rangle \quad \binom{8}{7} = 8 \text{ states, helicity } \lambda+\frac{7}{2}$$

$$b^1 b^2 b^3 b^4 b^5 b^6 b^7 b^8|\lambda\rangle \quad 1 \text{ state, helicity } \lambda+4$$

The only multiplet that avoids helicities higher than 2 and lower than -2 is as follows;

$$\lambda = -2: \left(-2, -\frac{3}{2}^8, -1^{28}, -\frac{1}{2}^{56}, 0^{70}, \frac{1}{2}^{56}, 1^{28}, \frac{3}{2}^8, 2\right)$$

This multiplet contains the graviton and thus, it is the supergravity multiplet. The CPT conjugate is not required. (justify and write about the massive states).
The most general algebra allowed by Lorentz invariance is (justify);

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = -2\delta^{AB} P_\mu \Gamma^\mu_{\alpha\beta} - 2iZ^{AB}\delta_{\alpha\beta}$$

where Z^{AB} have to be antisymmetric as we can show below. (Show).
If we have a massive particle, then we can go to the rest frame, and thus, $P_\mu = (-m, 0, ..., 0)$.
The algebra then becomes (derive this);

$$\{Q^A_\alpha, Q^{B\dagger}_\beta\} = 2\delta^{AB} m\delta_{\alpha\beta} + 2iZ^{AB}\Gamma_{\alpha\beta}$$

This implies the BPS bound (derive this properly and derive Z form);

$$m \geq q_i$$

So, the massless states have to be neutral also derive the short and ultrashort reps.

21.2.6 Supersymmetry in $d = 2$

In $d = 2$, we have the shortest spinor representation of size 1. It is a Majorana-Weyl representation and thus, we can use Weyl spinors for $d = 2$. To have $(N, \tilde{N})$ supersymmetry, we need N left moving and $\tilde{N}$ right moving fermions. To figure out the SUSY algebra in $d = 2$, we need to have Γ matrices in the left and right moving coordinates. derive them.
Now, the components of the Γ matrices are labeled by indices that can take the values (L, R).
So, the SUSY algebra is;

$$\{Q^A_L, Q^B_L\} = \delta^{AB}(P_0 - P_1), \ \{Q^A_R, Q^B_R\} = \delta^{AB}(P_0 + P_1), \ \{Q^A_L, Q^B_R\} = Z^{AB}$$

Is it still anti-symmetric?.

21.3 Differential forms and generalized gauge fields

tt

21.4 Thirty two supersymmetries

21.4.1 $d = 11$ supergravity

21.4.2 $d = 10$ II A supergravity

tt

21.4.3 $d = 10$ II B supergravity

tt

21.4.4 $d < 10$ supergravity

tt

21.5 Sixteen supersymmetries

tt

21.5.1 $d = 10, N = 1$ (type I) supergravity

tt

21.5.2 $d < 10$ supergravity

tt

21.5.3 $d = 6, N = 2$ supersymmetry

tt

21.5.4 $d = 4, N = 4$ supersymmetry

tt

21.6 Eight supersymmetries

tt

21.6.1 $d = 6, N = 1$ supersymmetry

tt

21.6.2 $d = 4, N = 2$ supersymmetry

tt

References

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[2] M. Bertolini, ''Lectures on supersymmetry,'' *Lecture notes given at SISSA*, 2015.