Best matching principle using Entropic Dynamics

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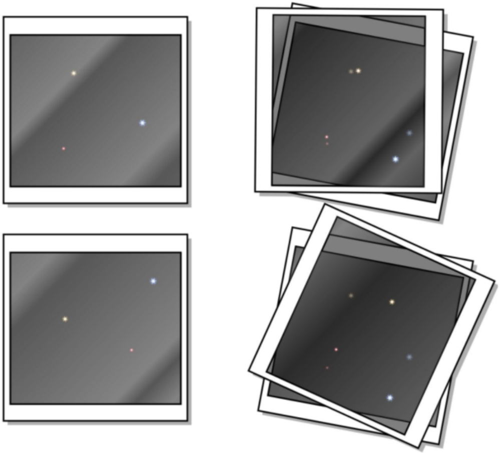
Equilocality

- Question of equilocality: "*What does it mean to have two objects at the same position at different times when only relational data is available?*"
- Newton and Leibniz knew about this problem.
- Newton proposed "absolute space" as the solution. Leibniz disagreed.

1. Best Matching
   - Equilocality
   - Barbour Bertotti Idea
2. Entropic Dynamics and Information Geometry
3. New Techniques
4. New works

Take two pictures of three stars



- Take tentative coordinates

$$\mathbf{r}_a(t_i) = r_a^i, \quad \mathbf{r}_a(t_f) = r_a^f$$

such that

$$||\mathbf{r}_a(t_i) - \mathbf{r}_b(t_i)|| = r_{ab}(t_i), \quad ||\mathbf{r}_a(t_f) - \mathbf{r}_b(t_f)|| = r_{ab}(t_f)$$

- Consider 9D vectors

$$q^i = \oplus \mathbf{r}_a^i = (x_1^i, y_1^i, z_1^i, x_2^i, y_2^i, z_2^i, x_3^i, y_3^i, z_3^i)$$

$$q^f = \oplus \mathbf{r}_a^f = (x_1^f, y_1^f, z_1^f, x_2^f, y_2^f, z_2^f, x_3^f, y_3^f, z_3^f)$$

- Natural distance (Euclidean distance)

$$d(q^i, q^f) = \left[\sum_{\alpha=1}^{9} (q_\alpha^f - q_\alpha^i)^2\right]^{1/2} = \left[\sum_{a=1}^{3} ||\mathbf{r}_a^f - \mathbf{r}_a^i||^2\right]^{1/2}$$

- Minimize the distance

$$d(q^i, q^{\text{BM}}) = \inf_{q'} d(q^i, q')$$

maintaining

$$||\mathbf{r}_a^{\text{BM}} - \mathbf{r}_b^{\text{BM}}|| = r_{ab}(t_f)$$

- For translations, best matching $\Rightarrow$ barycenters match (not center of mass).
- We can also do it for rotations.

1. Best Matching
2. Entropic Dynamics and Information Geometry
3. New Techniques
4. New works

Entropic Dynamics

- Take N particle system
- x_n^a are ontic variables $(n = 1, ..., N)$.
- $\rho(x|t)$ is the probability of $x = x_1^a, ..., x_N^a$ at time t.
- Probability changes as

$$\rho(x'|t') = \int dx\ P(x'|x, t, t')\rho(x|t)$$

- $P(x'|x, t, t')$ is determined by maximizing the relative entropy (relative to a prior Q) given some **constraints**

$$S[P, Q|\text{constraints}] = -P \ln \frac{P}{Q} - \sum_{\text{constraints}} (\text{Lagrange multipliers})(\text{constraints})$$

Entropic Dynamics

- Change in x_n^a

$$\hat{\Delta} x_n^a = ({x'}_n^a + {\xi'}_n^a) - (x_n^a + \xi_n^a) = \Delta x_n^a + \Delta \xi_n^a$$

- ${\xi'}_n^a$ and ξ_n^a are best matching transformations.
- Constraints;

$$\langle \delta_{ab} \hat{\Delta} x_n^a \hat{\Delta} x_n^b \rangle = \kappa_n \quad \text{(No correlation)}$$

$$\sum_n \langle \Delta x_n^a \rangle \frac{\partial \varphi}{\partial x_n^a} = \kappa' \quad \text{(Correlations)}$$

Entropic Dynamics

- $P(x'|x)$ (conditional probability);

$$P(x'|x) = \frac{1}{Z} \exp\left[-\sum_n \frac{\alpha_n}{2} \delta_{ab} (\hat{\Delta} x_n^a - \overline{\Delta x_n^a})(\hat{\Delta} x_n^b - \overline{\Delta x_n^b})\right]$$

$$\text{where} \quad \overline{\Delta x_n^a} = \frac{\alpha'}{\alpha_n} \delta^{ab} \partial_{nb} \varphi - \Delta \xi_n^a$$

Entropic Dynamics

- Relating α_n and α' with Δt

$$\alpha' = \frac{1}{\hbar} = \text{constant}, \quad \alpha_n = \frac{m_n}{\hbar \Delta t}$$

- A new index

$$A = (a, n)$$

- Introduce mass tensor

$$m_{AB} = m_n \delta_{ab}, \quad m^{AB} = \frac{1}{m_n} \delta_{ab}$$

Entropic Dynamics

- Dynamical equation of $\rho(x,t)$ (Fokker Planck equation)

$$\partial_t \rho(x,t) + \partial_A \left[\rho(x,t) v^A(x,t)\right] = 0$$

where

$$v^A = m^{AB}\partial_B \phi - \dot{\xi}^A, \quad \dot{\xi}^A = \frac{\Delta \xi^A}{\Delta t}, \quad \phi = \hbar(\varphi - \ln\sqrt{\rho})$$

- Hamiltonian form

$$\partial_t \rho = \frac{\partial \tilde{H}}{\partial \phi} \text{ with } \tilde{H}[\rho,\phi] = \frac{1}{2}\int dx\, \rho\, m^{AB}\left(\partial_A \phi - \dot{\xi}_A\right)\left(\partial_B \phi - \dot{\xi}_B\right) + F[\rho]$$

- The second equation (HJ type equation)

$$-\partial_t \phi = m^{AB}(\partial_A \phi - m_{AC}\dot{\xi}^C)(\partial_B \phi - m_{BD}\dot{\xi}^D) + \frac{\partial F[\rho]}{\partial \rho}$$

The metric

- Take a statistical manifold S;
  - Parameters $\theta_i = \theta_1, ..., \theta_N$
  - Possible values of variables (Sample space) X
- The metric on S (Fischer-Rao);

$$g_{ij}(\theta) = \int_X p(x|\theta) \frac{\partial \ln p(x|\theta)}{\partial \theta_i} \frac{\partial \ln p(x|\theta)}{\partial \theta_j} \, dx$$

Previous work (2016)

- Define joint probability

$$\rho(x, x'|t, t') = P(x'|x, t, t')\rho(x|t)$$

- Distance between $\rho(x', x|t' + dt, t)$ and $\rho(x', x|t', t)$ is (long calc.)

$$dT^2 = G\, dt^2 \text{ where } G = \tilde{H}_0[\rho, \phi] + (\xi \text{ independent terms})$$

and

$$\tilde{H}_0[\rho, \phi] = \int dx \left[\frac{m^{AB}}{2}\rho(\partial_A\phi - \dot{\xi}_A)(\partial_B\phi - \dot{\xi}_B) + \frac{\hbar^2}{8\rho}\partial_A\rho\partial^A\rho\right]$$

Previous work (2016)

- Minimizing G, we get

$$M\dot{\xi}_a^{BM} = -\int dx\, \rho \sum_n \frac{\partial\phi}{\partial x_n^a} = -\int dx\, \rho \frac{\partial\phi}{\partial X^a} = -\langle P_a \rangle$$

  where P_a is the momentum of the center of mass.

- This means that

$$\Delta\xi_a^{\text{BM}} = -\Delta t \frac{\langle P_a \rangle}{M}$$

New techniques

- The e-configuration space is;

$$S = \left\{ \rho(x) = \rho_x \middle| \int dx\ \rho(x) = 1 \right\}$$

- Introduce the cotangent space $T^*S = \{(\rho^x, \phi_x)\} = (X^{1x}, X^{2x}) = X^{\alpha x}$.
- It has a natural symplectic form

$$\Omega = \int dx (\tilde{\nabla}\rho_x \otimes \tilde{\nabla}\phi_x - \tilde{\nabla}\phi_x \otimes \tilde{\nabla}\rho_x) = -\tilde{d}\left(\underbrace{\int dx \phi_x \tilde{d}\rho^x}_{\theta} \right) \Rightarrow \Omega_{\alpha x, \beta x'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta_{xx'}$$

which is closed $(\tilde{d}\Omega = 0)$ and locally exact $(\Omega = -\tilde{d}\theta)$

New techniques

- Symplectic structure should be preserved

$$\mathcal{L}_H \Omega = 0 \Rightarrow \frac{d\rho_x}{d\lambda} = \frac{\partial \tilde{H}}{\partial \phi^x}, \quad \frac{d\phi^x}{d\lambda} = -\frac{\partial \tilde{H}}{\partial \rho_x}$$

  which are Hamilton's equations $\rightarrow$ Hamiltonian flows.

- Normalization preserved i.e.

$$\{N, \tilde{H}\} = 0 \text{ with } N = 1 - \int dx\ \rho_x$$

- The coordinates change as follows;

$$\rho_x(\nu) = \rho_x(0), \quad \phi_x(\nu) = \phi_x(0) + \nu$$

Metric

- Summary of the procedure

$$\underbrace{g_{xx'} = A(|\rho|)n_x n_{x'} + \frac{\hbar}{2\rho_x}\delta_{xx'}}_{S} \xrightarrow{\text{natural guess}} \underbrace{g_{xx'}d\rho^x d\rho^{x'} + g^{xx'}d\phi_x d\phi_{x'}}_{T^*S^+}$$

$$\xrightarrow{|\rho|=1} \underbrace{\frac{\hbar}{2\rho^x}(d\rho^x)^2 + \frac{2\rho^x}{\hbar}((d\phi_x)^2 - n_x\langle d\phi\rangle)}_{T^*S} \text{ where } \langle d\phi\rangle = \int dx\ \rho^x d\phi_x$$

- The final metric is (FS metric)

$$G_{\alpha x,\beta x'}dX^{\alpha x}dX^{\beta x'} = \frac{\hbar}{2\rho^x}(d\rho^x)^2 + \frac{2\rho^x}{\hbar}((d\phi_x)^2 - n_x\langle d\phi\rangle)$$

Hamilton-Killing flows

- Define new coordinates;

$$\psi_x = \sqrt{\rho_x} e^{i\phi_x/\hbar}, \;\; \psi_x^* = \sqrt{\rho_x} e^{-i\phi_x/\hbar}$$

- Under normalization flow

$$\psi_x \to e^{i\nu/\hbar}\psi_x, \;\; \psi_x^* \to e^{-i\nu/\hbar}\psi_x^*$$

- Metric structure is preserved (Killing flows)

$$\mathcal{L}_H G = 0$$

Schrodinger equation

- Impose $\mathcal{L}_H G = 0$

$$\frac{\partial \tilde{H}}{\partial \psi_x \partial \psi_{x'}} = \frac{\partial \tilde{H}}{\partial \psi^*_x \partial \psi^*_{x'}} = 0 \xrightarrow{\text{Norm argument}} \tilde{H} = \int dx \int dx' \; \psi^*_x H_{xx'} \psi_{x'}$$

- $\mathcal{L}_H \Omega = 0 \rightarrow$ Hamilton's equations which become

$$i\hbar \frac{d\psi_x}{d\tau} = \int dx \; H_{xx'} \psi_{x'}, \quad i\hbar \frac{d\psi^*_x}{d\tau} = -\int dx \; \psi^*_{x'} H_{xx'}$$

i.e. the Schrodinger equation (possibly nonlocal Hamiltonian).

- To reproduce Fokker Planck equation

$$\tilde{H}_\xi = \int dx \; \psi^* \left(\frac{1}{2} m^{AB} (i\hbar\partial_A - \dot{\xi}_A)(i\hbar\partial_B - \dot{\xi}_B) + V(x_n) \right) \psi$$

Translations

- Metric

$$ds^2 = 4\hbar(1 - |\langle\psi|\psi'\rangle|)$$

- Take $\psi' = \psi + \delta\psi$

$$ds^2 = 2\hbar|\langle\psi|\delta\psi\rangle| + \text{Other terms}$$

- ψ on different time slices

$$\psi_{t+dt} = \psi_t + \delta_\xi\psi \Rightarrow ds^2 = 2\hbar|\langle\psi|\delta_\xi\psi\rangle|$$

- where

$$i\hbar|\delta_\xi\psi\rangle = dt\ H_\xi|\psi\rangle$$

Translations

- Minimize ds^2 w.r.t $\dot{\xi}_a$

$$\frac{\partial \tilde{H}_\xi[\psi,\psi^*]}{\partial \dot{\xi}_a} = 0$$

where

$$\tilde{H}_\xi = \int dx\; \psi^* \sum_n \left(\frac{1}{2m_n} \delta^{bc} (i\hbar\partial_{nb} - m_n\dot{\xi}_b)(i\hbar\partial_{nc} - m_n\dot{\xi}_c) + V(x_n) \right) \psi$$

$$\Rightarrow \dot{\xi}_a^{\text{BM}} = -\frac{\langle\psi|P_a|\psi\rangle}{M} = -\frac{\langle P_a\rangle}{M} \Rightarrow \Delta\xi_a^{\text{BM}} = -\Delta t \frac{\langle P_a\rangle}{M}$$

Rotations

- We now have

$$x_n^a \to x_n^a + i\zeta_k (J^k)^a{}_b x^b$$

$$\Rightarrow \xi_n^a = i\zeta_k (J^k)^a{}_b x^b \quad (J^k \text{ are } SO(3) \text{ generators})$$

- The Hamiltonian becomes

$$H_{\dot\zeta} = -\frac{1}{2}\dot\zeta_k \dot\zeta_l \langle I^{kl}\rangle - \dot\zeta_k \sum_n \langle (x_n \times \hat{P}_n)^k \rangle + \text{Others}$$

where

$$I^{kl} = \sum_n m_n (g^{kl} ||x_n||^2 - x_n^k x_n^l)$$

- Minimizing w.r.t $\dot\zeta_k$ (best matching) gives

$$\dot\zeta_k^{\text{BM}} \langle I^{kl}\rangle = -\langle L^l_{\text{tot}}\rangle$$

Dilations

- For dilations, we have

$$x_n^a \to x_n^a + \dot{\lambda} x_n^a \Rightarrow \xi_n^a = \dot{\lambda} x_n^a$$

- to get the Hamiltonian

$$H_{\dot{\lambda}} = \dot{\lambda}^2 \sum_n \frac{m_n}{2} \langle ||x_n||^2 \rangle + \dot{\lambda} \sum_n \langle x_n . P_n \rangle + \text{Others}$$

- Minimizing, we get

$$\dot{\lambda}^{\text{BM}} \sum_n m_n \langle ||x_n||^2 \rangle = - \sum_n \langle x_n^a P_{na} \rangle$$

Special Conformal Transformations (SCTs)

- For SCT, we have;

$$x_n^a \to x_n^a + 2(x_n^b \dot{l}_b)x_{an} - ||x_n||^2 \dot{l}_a \Rightarrow \dot{\xi}_n^a = 2(x_n^b \dot{l}_b)x_n^a - ||x_n||^2 \dot{l}^a$$

- to get the Hamiltonian

$$H_{\dot{l}} = \frac{1}{2}\dot{l}^2 \sum_n m_n \langle ||x_n||^2 \rangle + \dot{l}^a \sum_n \langle 2(x_n^T.P_n)x_{na} - ||x_n||^2 P_{na} \rangle + \text{Others}$$

- Minimizing, we get;

$$(\dot{l}^{\text{BM}})^b \sum_n m_n \langle ||x_n||^4 \rangle = -\sum_n \langle (2x_n^a x_n^b - ||x_n||^2 g^{ab}) P_{na} \rangle$$

General Case

- General transformations

$$x_n^a \to x_n^a + \dot{\omega}_l \pi^l_{ab} x_n^b \Rightarrow \dot{\xi}_n^a = \dot{\omega}_l \pi^l_{ab} x_n^b$$

- The Hamiltonian is **a bit complicated to write here**.
- Minimizing, we get;

$$\dot{\omega}_k^{\mathsf{BM}} \sum_n m_n \langle X_n^{(k)T} . X_n^{(l)} \rangle = - \sum_n \langle X_n^{(l)T} . P_n \rangle$$

where

$$X_n^{(k)} = \pi^k_{ab} x_n^b$$

Adding a gauge field

- It will need another constraint;

$$\langle \hat{\Delta} x_n^a \rangle A_a = \kappa_n''$$

- The only significant change

$$\partial_{nb}\phi \to \partial_{nb}\phi - \hbar\beta_n A_a = D_{nb}\phi$$

which implies

$$\partial_{na}\psi \to \partial_{na}\psi - i\beta_n A_a\psi = D_{na}\psi \quad \left(\beta_n = \frac{q_n}{c\hbar}\right)$$

which further implies

$$P_{na}\psi = -i\hbar\partial_{na}\psi \to P_{na}\psi - \hbar\beta_n A_a\psi = \mathbb{P}_{na}\psi$$