Constraining boundary conditions in non-rational CFTs

(Based on 2504.00367)

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Motivation

Motivation

- Non-rational CFTs are ubiquitous (e.g. the free boson)
- RCFT methods to determine boundary states don't always translate to non-rational CFTs.
- We may need different methods for non-rational CFTs.
- We study free boson boundary states as a lab to study the distinctions between these methods

Free boson

Boundary states

- The boundary conditions of 2D CFT on a boundary $\leftrightarrow$Boundary states $||B_\alpha\rangle\rangle$.
- A CFT contains primary fields ϕ_i $\leftrightarrow$ primary states $|i\rangle$.
- Spin zero primary states $|i\rangle$ $\leftrightarrow$ Ishibashi states $||i\rangle\rangle$.
- Constructing $||B_\alpha\rangle\rangle$.

$$||B_\alpha\rangle\rangle = \sum_i A_{\alpha i}||i\rangle\rangle$$

Compact free boson

- Compact boson $\rightarrow X \sim X + 2\pi R$
- Classical solution $(\alpha' = 1)$

$$X(z,\bar{z}) = x_0 - \frac{i}{2}(p_L \ln z + p_R \ln \bar{z}) + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} \left(a_n^\dagger z^{-n} + a_n z^n + \tilde{a}_n^\dagger \bar{z}^{-n} + \tilde{a}_n \bar{z}^n \right)$$

$$= X_L(z) + X_R(\bar{z})$$

where

$$p_L = \frac{n}{R} + mR \quad p_R = \frac{n}{R} - mR \quad (n, m \in \mathbb{Z})$$

Ishibashi states for free bosons

- Primary fields (type 1)

$$\mathcal{V}_{n,m}(z,\bar{z}) =: e^{ik_L X_L(z) + ik_R X_R(\bar{z})} :$$

$$h = \frac{1}{4}\left(\frac{n}{R} + mR\right)^2 \quad \bar{h} = \frac{1}{4}\left(\frac{n}{R} - mR\right)^2$$

- Ishibashi states ($h = \bar{h}$)

$$\|(n,0)\rangle\!\rangle \quad \|(0,m)\rangle\!\rangle$$

- Primary fields (type 2)

$$\mathcal{V}_{[J,J]}(z,\bar{z}) = \mathcal{N}_J V_J(z)\bar{V}_J(\bar{z})$$

where

$$V_J(z) = \left(\oint \frac{du}{2\pi} : e^{-2iX_L(z+u)} :\right) : e^{2iX_L(z)} :$$

- The conformal weights

$$h(V_{[J,J]}) = \bar{h}(V_{[J,J]}) = J^2$$

- Ishibashi states

$$\|[J,J]\rangle\!\rangle$$

Boundary states for free bosons

- Boundary states, at any radius[1]

$$\|D;x_0\rangle\rangle = \frac{1}{\sqrt{\sqrt{2}R}}\left(\sum_{J=0}^{\infty}\|[J,J]\rangle\rangle + \sum_{n\neq 0} e^{-inx_0/R}\|(n,0)\rangle\rangle\right)$$

$$\|N;\tilde{x}_0\rangle\rangle = \sqrt{\frac{R}{\sqrt{2}}}\left(\sum_{J=0}^{\infty}(-1)^J\|[J,J]\rangle\rangle + \sum_{m\neq 0} e^{-imR\tilde{x}_0}\|(0,m)\rangle\rangle\right)$$

[1] See for example M. Oshikawa and I. Affleck 1997

- Friedan-Janik (FJ) boundary state (at any radius)[2]

$$||F(cos\theta)\rangle\rangle = \mathcal{C}(\theta) \sum_{J=0}^{\infty} P_J(\cos\theta)||[J,J]\rangle\rangle \quad (-1 \leq cos\theta \leq 1)$$

where

$P_J(x)$ are Legendre polynomials

[2] R. Janik 2001

Problems with FJ states

Problem 1

- Continuous open string spectrum[3]

$$\langle\!\langle F(\cos\theta_1)|q^H|F(\cos\theta_2)\rangle\!\rangle = \bar{\mathcal{C}}(\theta_1)\mathcal{C}(\theta_2)\sum_{J=0}^{\infty}P(\cos\theta_1)P(\cos\theta_2)\frac{q^{J^2}-q^{(J+1)^2}}{\eta(q)}$$

$$= \int_0^{\infty} dh\, \rho(h)\chi_h(\tilde{q})$$

- We calculated this $\rho(h)$ (coming up)

[3] R. Janik 2001

- Compare it with the open spectra of Dirichlet and Neumann states

$$\langle\langle D; x_0 | q^H | D; x_0' \rangle\rangle = \sum_{m \in \mathbb{Z}} \chi_{R^2\left(m - \frac{x_0 - x_0'}{2\pi R}\right)^2}(\tilde{q})$$

$$\langle\langle N; \tilde{x}_0 | q^H | N; \tilde{x}_0' \rangle\rangle = \sum_{m \in \mathbb{Z}} \chi_{\frac{1}{R^2}\left(m + \frac{R(x_0 - x_0')}{2\pi}\right)^2}(\tilde{q})$$

- Discrete open string spectra here

Problem 2

- Cluster condition

$$B_{\alpha i} B_{\alpha j} = \sum_k M_{ij}^k B_{\alpha k}$$

where

$$B_{\alpha i} = \frac{A_{\alpha i}}{A_{\alpha 0}} \qquad M_{ij}^k = C_{ij}^k F_{k0} \begin{bmatrix} j & j \\ i & i \end{bmatrix}$$

- FJ states satisfy cluster condition for $i = \|[J,J]\rangle\rangle$, $j = \|[J',J']\rangle\rangle$. **What about other primaries?**

New Ishibashi states

- Derive new Ishibashi states

$$\|(n,0)\rangle\rangle_{+} = \frac{1}{\sqrt{2}}\left(\|(n,0)\rangle\rangle + \|(-n,0)\rangle\rangle\right)$$

$$\|(n,0)\rangle\rangle_{-} = \frac{1}{i\sqrt{2}}\left(\|(n,0)\rangle\rangle - \|(-n,0)\rangle\rangle\right)$$

$$\|(0,m)\rangle\rangle_{+} = \frac{1}{\sqrt{2}}\left(\|(0,m)\rangle\rangle + \|(0,-m)\rangle\rangle\right)$$

$$\|(0,m)\rangle\rangle_{-} = \frac{1}{i\sqrt{2}}\left(\|(0,m)\rangle\rangle - \|(0,-m)\rangle\rangle\right)$$

M_{ij}^k coefficients

- Dirichlet and Neumann states satisfy the cluster condition
- Use this fact to derive M_{ij}^k coefficients
- Some relevant M_{ij}^k coefficients[4]

$$\sum_{J=0}^{\infty} M^{[J,J]}_{(n,0)_+(n,0)_+} = \sum_{J=0}^{\infty} M^{[J,J]}_{(0,m)_+(0,m)_+} = 1$$

$$\sum_{J=0}^{\infty} M^{[J,J]}_{(n,0)_-(n,0)_-} = \sum_{J=0}^{\infty} M^{[J,J]}_{(0,m)_-(0,m)_-} = 1$$

[4] Y. Cai, D. Robbins, HS (to appear)

M_{ij}^k coefficients from FJ states

- If FJ states satisfy cluster condition, then we have

$$\sum_{J=0}^{\infty} M^{[J,J]}_{(n,0)_\pm (n,0)_\pm} P_J(\cos\theta) = 0$$

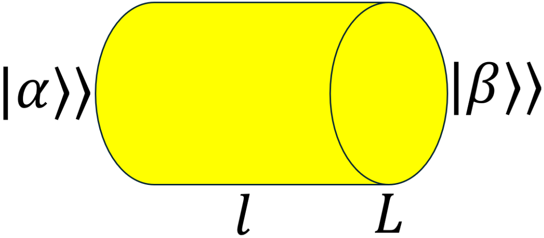
$$\sum_{J=0}^{\infty} M^{[J,J]}_{(0,m)_\pm (0,m)_\pm} P_J(\cos\theta) = 0$$

- Using orthogonality of $P_J(\cos\theta)$ we get

$$M^{[J,J]}_{(n,0)_+ (n,0)_+} = M^{[J,J]}_{(0,m)_+ (0,m)_+} = 0$$

$$M^{[J,J]}_{(n,0)_- (n,0)_-} = M^{[J,J]}_{(0,m)_- (0,m)_-} = 0$$

Problem 3: g function



- Partition function on cylinder with length l, circumference L with boundary conditions α and β

$$Z_{\alpha\beta}(l, L) = \langle\langle\alpha|e^{-lH}|\beta\rangle\rangle$$

- Long cylinder limit

$$\lim_{l\to\infty} Z_{\alpha\beta}(l,L) = g_\alpha g_\beta e^{-lE_0}$$

- g_α is g function of $\|\alpha\rangle\rangle \to$ boundary entropy of $\|\alpha\rangle\rangle$

g function examples

- g function for $||N;\alpha\rangle\rangle$, $||D;\beta\rangle\rangle$ and $||F(\cos\theta)\rangle\rangle$ states

$$g_{N(\alpha)} = \sqrt{\frac{R}{\sqrt{2}}}$$

$$g_{D(\beta)} = \frac{1}{\sqrt{\sqrt{2}R}}$$

$$g_{F(\theta)} = \mathcal{C}(\theta)$$

Gaberdiel Recknagel states

- Gaberdiel Recknagel (GR) states are at rational multiples of self-dual radius

$$R = \frac{M}{N} \quad (M, N \in \mathbb{Z}_{>0})$$

- States[5]

$$\|g\rangle\rangle_{M,N} = \frac{\sqrt{MN}}{\sqrt[4]{2}} \sum_{j,m,n} (P_N^+ P_M^-(g))^j_{m,n} \|j, m, n\rangle\rangle$$

  where $g \in SU(2)$ and P_N^+, P_M^- are projection operators.

- g function of $\|g\rangle\rangle_{M,N}$ states

$$g_{g,M,N} = \frac{\sqrt{MN}}{\sqrt[4]{2}}$$

[5] M. Gaberdiel, A. Recknagel (2001)

- For any real R, there exist sequences $\{M_1, M_2, ...\}$ and $\{N_1, N_2, ...\}$ such that

$$\lim_{k\to\infty} \frac{M_k}{N_k} = R$$

- For irrational R, the sequences $\{M_1, M_2, ...\}$ and $\{N_1, N_2, ...\}$ diverge.

- GR states in irrational R limit become Friedan states. Write $g \in SU(2)$ as

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \text{ with } |a|^2 + |b|^2 = 1$$

then we get

$$\lim_{M,N\to\infty} ||g\rangle\rangle_{M,N} = ||F(\cos\theta)\rangle\rangle \text{ with } \cos\theta = 2|a|^2 - 1$$

- This implies the following

$$g_{F(\theta)} = \lim_{M,N\to\infty} g_{g,M,N}$$

$$\Rightarrow \mathcal{C}(\theta) = \lim_{M,N\to\infty} \frac{\sqrt{MN}}{\sqrt[4]{2}} = \infty$$

- The normalization parameter diverges!

Density of states

Density of states $\rho(h)$

- We saw that

$$\langle\langle F(\cos\theta_1)|q^H|F(\cos\theta_2)\rangle\rangle = \int_0^\infty dh\, \rho(h)\chi_h(\tilde{q})$$

- What is this $\rho(h)$?

- Define three regions for every $n \in \mathbb{Z}_{\geq 0}$

$$\mathrm{I}_n : \left[\frac{n^2}{4}, \frac{1}{4}\left(n + \frac{\theta_2 - \theta_1}{2\pi}\right)^2\right) \cup \left[\frac{1}{4}\left(n + 1 - \frac{\theta_2 - \theta_1}{2\pi}\right)^2, \frac{1}{4}(n+1)^2\right)$$

$$\mathrm{II}_n : \left[\frac{1}{4}\left(n + \frac{\theta_2 - \theta_1}{2\pi}\right)^2, \frac{1}{4}\left(n + \frac{\theta_2 + \theta_1}{2\pi}\right)^2\right) \cup \left[\frac{1}{4}\left(n + 1 - \frac{\theta_2 + \theta_1}{2\pi}\right)^2, \frac{1}{4}\left(n + 1 - \frac{\theta_2 - \theta_1}{2\pi}\right)^2\right)$$

$$\mathrm{III}_n : \left[\frac{1}{4}\left(n + \frac{\theta_2 + \theta_1}{2\pi}\right)^2, \frac{1}{4}\left(n + 1 - \frac{\theta_2 + \theta_1}{2\pi}\right)^2\right)$$

- The expression for $\rho(h)$

$$\rho(h) = \begin{cases} 0, & \text{region } \mathsf{I}_n \\ \frac{2\sqrt{2}\,\overline{\mathcal{C}(\theta_1)}\mathcal{C}(\theta_2)}{\pi\sqrt{h}} \sqrt{\frac{1-\cos(4\pi\sqrt{h})}{\cos(4\pi\sqrt{h})-\cos(\theta_1+\theta_2)}}\, K\left(-\frac{\cos(\theta_2-\theta_1)-\cos(4\pi\sqrt{h})}{\cos(4\pi\sqrt{h})-\cos(\theta_1+\theta_2)}\right) & \text{region } \mathsf{II}_n \\ \frac{2\sqrt{2}\,\overline{\mathcal{C}(\theta_1)}\mathcal{C}(\theta_2)}{\pi\sqrt{h}} \sqrt{\frac{1-\cos(4\pi\sqrt{h})}{\cos(\theta_1+\theta_2)-\cos(4\pi\sqrt{h})}}\, K\left(-\frac{\cos(\theta_2-\theta_1)-\cos(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)-\cos(4\pi\sqrt{h})}\right) & \text{region } \mathsf{III}_n \end{cases}$$

where

$$K(m) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-mu^2)}}.$$

is the complete elliptic integral of the first kind.

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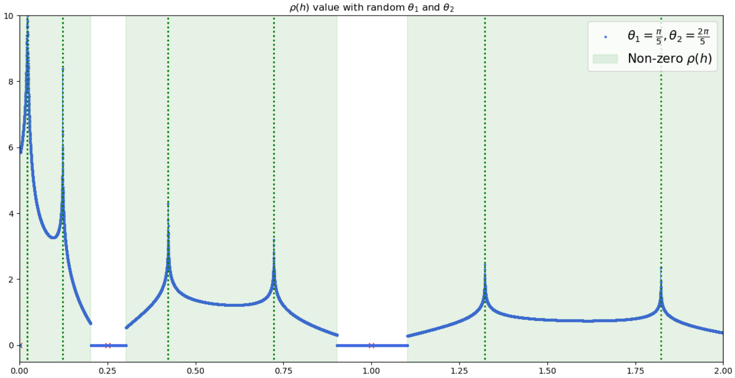
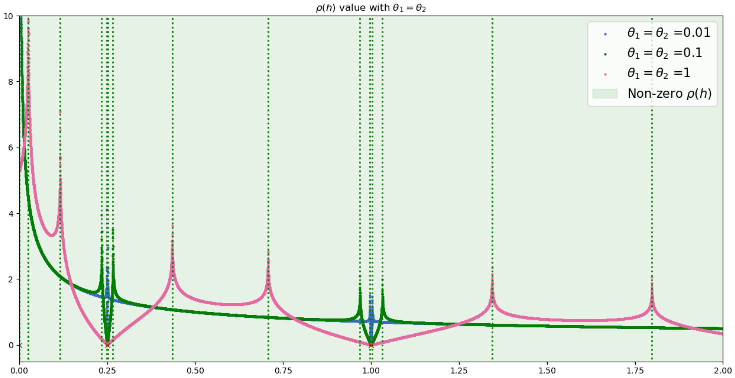
$\rho(h)$ for typical θ_1, θ_2



$\rho(h)$ for $\theta_1 = \theta_2$



$\rho(h)$ for $\theta_1 = \theta_2$

- For $\theta_1 = \theta_2 = \epsilon$

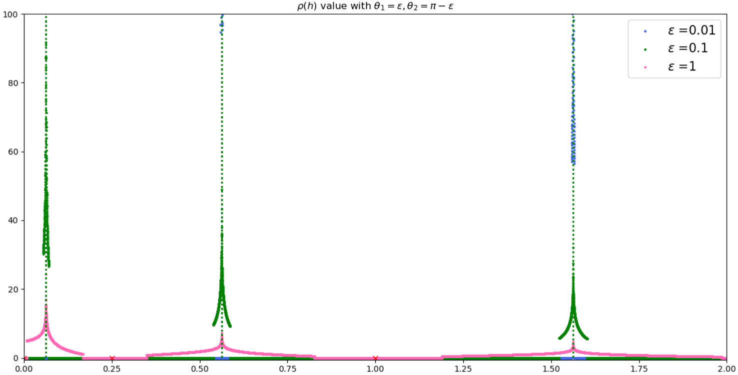
$$\lim_{\epsilon\to 0} \rho(h) = \frac{\sqrt{2}|\mathcal{C}(1)|^2}{\sqrt{h}} + \sum_n c_n \delta\left(h - \frac{n^2}{4}\right)$$

but

$$\int_0^\infty dh \sum_n c_n \delta\left(h - \frac{n^2}{4}\right) = 0$$

$\rho(h)$ for $\theta_2 = \pi - \theta_1$

- For $\theta_2 = \pi - \theta_1$, we have



Thanks for listening
Questions?